Asymptotic of solutions of friction type differential equations disturbed by stable Lévy noise

Richard Eon

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Richard Eon

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Institut de Recherche Mathématique de Rennes
UFR de mathématiques

Asymptotique des solutions d'équations différentielles de type frottement perturbées par des bruits de Lévy stables

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devant le jury composé de :

Nicolas FOURNIER
Professeur des Universités, LPMA / rapporteur

Anthony RÉVEILLAC
Professeur des Universités, INSA Toulouse / rapporteur

Loïc CHAUMONT
Professeur des Universités, Angers / examinateur

Nathalie KRELL
Maître de conférences, Rennes 1 / examinateur

Mihai GRADINARU
Professeur des Universités, Rennes 1 / directeur de thèse

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Contents

Introduction

Motivation et modèle

Dans cette thèse, nous allons étudier des équations différentielles qui caractérisent le déplacement d'un objet, dont la position et la vitesse sont des éléments de $\mathbb{R}^d$, soumis à une force de frottement :

$$\begin{cases} v'_t = -\frac{1}{2}\nabla\mathcal{U}(v_t), \\ x'_t = v_t \end{cases}$$

où $\mathcal{U}$ est une fonction de $\mathbb{R}^d$ dans $\mathbb{R}^d$ appelée potentiel et où les conditions initiales seront précisées plus tard.

L'action d'une force de frottement introduit deux hypothèses sur $\mathcal{U}$. Tout d'abord, si la vitesse de l'objet est nulle, il n'y a pas de force de frottement, ce qui se traduit par $\nabla\mathcal{U}(0) = 0$. Deuxièmement, comme une force de frottement est toujours opposée au vecteur vitesse de l'objet, il existe $b : \mathbb{R}^d \to \mathbb{R}^+$ telle que, pour tout $v \in \mathbb{R}^d$, si $||v||$ désigne la norme euclidienne de v, $\frac{1}{2}\nabla\mathcal{U}(v) = \frac{v}{||v||}b(v)$. De plus, $b(v) = 0 \Leftrightarrow v = 0$. De ces deux hypothèses découlent deux propriétés fondamentales de ces équations :

- 0 est un point stable donc la fonction vitesse identiquement nulle est solution de ces équations ;
- 0 est un point attractif donc si v est une solution de ces équations alors $\lim\limits_{t\to+\infty} v_t = 0$.

Donnons ici quelques exemples de fonctions classiques que l'on peut trouver.

1) En théorie cinétique classique, les forces de frottement sont proportionnelles à la vitesse donc b est de la forme $b(v) = k||v||$ avec $k > 0$.
2) En dynamique des fluides, Lord Rayleigh a introduit une force visqueuse quadratique en considérant le cas $b(v) = k||v||^2$.
3) En aérodynamique, on étudie parfois le modèle où $b(v) = k||v||^3$.
4) De façon générale, on peut considérer tous les b de la forme $b(v) = k||v||^\beta$ avec $\beta \geq 1$.
5) Le cas $b = \mathbb{1}_{\mathbb{R}\setminus\{0\}}$ est également intéressant puisqu'il correspond au cas où la force est constante mais toujours opposée au mouvement.

En 1908, Langevin introduit une équation où le bilan des forces contient également une force de perturbation, résultante des nombreux chocs subis par l'objet, qui sera donc supposée aléatoire et plus exactement brownienne :

$$\begin{cases} \mathrm{d}v_t = -v_t\mathrm{d}t + \mathrm{d}b_t, \\ \mathrm{d}x_t = v_t\mathrm{d}t \end{cases}$$

où b est un mouvement brownien standard.

Cette équation peut être généralisée de deux façons différentes. Tout d'abord, cette équation est écrite pour une force de frottement linéaire mais elle peut être écrite pour une force de frottement quelconque :

$$\begin{cases} \mathrm{d}v_t = -\frac{1}{2}\mathcal{U}'(v_t)\mathrm{d}t + \mathrm{d}b_t, \\ \mathrm{d}x_t = v_t\mathrm{d}t. \end{cases}$$

Ensuite, dans cette équation, la perturbation est supposée continue mais de récents travaux, concernant certains systèmes physiques, en chimie, en biologie, en économie ou encore en finance s'intéressent à des perturbations avec des sauts de type Lévy. Citons par exemple l'étude de l'élargissement spectral dans les plasmas, les articles de A. Chechkin, J. Klafter, V. Gonchar, R. Metzler et L. Tanatrov : [9] et [10], ou encore les systèmes avec interactions longues portées qui étudient le comportement des albatros (Lévy flights). On peut également citer les travaux de Barndorff-Nielsen et Shephard dans [5] sur les modèles de volatilité en finance. Nous considérons donc le modèle où la perturbation ℓ est un processus de Lévy :

$$\begin{cases} \mathrm{d}v_t = -\frac{1}{2}\mathcal{U}'(v_t)\mathrm{d}t + \mathrm{d}\ell_t, \\ \mathrm{d}x_t = v_t\mathrm{d}t. \end{cases}$$

Résultats principaux

Dans la première partie de cette thèse, nous rappelons les propriétés des processus de Lévy qui permettent de donner un sens à cette équation. En particulier, nous souhaitons étudier le comportement de la famille de solutions des eds suivantes

$$\begin{cases} \mathrm{d}v_t^\varepsilon = -\mathrm{sgn}(v_t^\varepsilon)|v_t^\varepsilon|^\beta\mathrm{d}t + \varepsilon\mathrm{d}\ell_t, \\ \mathrm{d}x_t^\varepsilon = v_t^\varepsilon\mathrm{d}t \end{cases}$$

où $\beta \in \mathbb{R}$ et ℓ est un processus de Lévy α-stable avec $\alpha \in (0, 2]$. On adoptera la convention $\mathrm{sgn}(0) = 0$ pour que 0 soit un point fixe. Dans un premier temps, on démontre qu'il existe une unique solution globale pour chacune de ces équations différentielles stochastiques. Pour $\alpha = 2$, le processus est un mouvement brownien standard et ces équations différentielles stochastiques font l'objet de nombreuses études, nous pouvons par exemple citer les ouvrages de [36], [22] et [37]. Dans le cas $\beta \geq 1$, la dérive est localement lipschitzienne ce qui implique l'existence et l'unicité d'une solution locale. Pour $\beta \in (-1, 1)$, on peut obtenir le même résultat par le théorème de Girsanov. Pour montrer que la solution est globale, on peut utiliser le théorème basé sur la fonction d'échelle et la mesure vitesse que l'on peut trouver par exemple dans [23] ou dans [37]:

Théorème. *Supposons que X est solution de l'équation intégrale stochastique*

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s$$

où σ ne s'annule pas et telle que $b\sigma^{-2}$ est localement intégrable. On définit alors la fonction d'échelle

$$s(x) := \int_0^x \exp\left(-2\int_0^y b(z)\sigma^{-2}(z)\mathrm{d}z\right)\mathrm{d}y$$

et la mesure vitesse

$$m(\mathrm{d}x) = s'^{-1}(x)\sigma^{-2}(x)\mathrm{d}x.$$

Alors une condition suffisante de non explosion est

$$\int_0^{+\infty} s'(x)\int_0^x m(\mathrm{d}y)\mathrm{d}x = \infty.$$

Dans le cas $\alpha < 2$, on ne peut plus utiliser cet argument. Toutefois, dans le cas $\beta \geq 1$ où la dérive est localement lipschitzienne, l'étude des eds dirigées par des processus de Lévy faite dans [1] permet de démontrer l'existence et l'unicité d'une solution locale. Pour $\beta \in (0,1)$, la dérive n'est plus localement lipschitzienne mais elle reste continue et attractive ; le théorème 170 dans [41] permet alors de montrer l'existence et l'unicité de la solution. Pour étudier la non explosion de ces solutions, on démontre la proposition suivante :

Proposition. *Supposons que X est solution de*

$$X_t = X_0 - \int_0^t f(X_s)\mathrm{d}s + \varepsilon\ell_t$$

où ℓ est un processus de Lévy α-stable avec $\alpha < 2$ et $\varepsilon > 0$. Supposons que f est localement lipschitzienne, $f(0) = 0$, f est croissante et $\lim_{y\to\pm\infty} f(y) = \pm\infty$. Alors il existe une unique solution globale et, de plus, pour tout $\delta \in (0,\alpha)$ et tout $T > 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^\delta\Big] < \infty.$$

La preuve de cette proposition repose sur les propriétés des processus α-stables comme la décomposition d'Itō-Lévy que l'on peut retrouver dans [40] et sur un procédé d'entre-lacement décrit par exemple dans [15]. En effet, nous considérons tout d'abord la solution de l'équation où le processus de Lévy ne possède plus de "grands" sauts et nous montrons que dans ce cas la proposition est vérifiée. Nous montrons ensuite que la contribution des "grands" sauts ne peut pas faire exploser le processus. Dans le deuxième chapitre, nous nous intéressons au comportement de la solution lorsque la perturbation aléatoire est symétrique et "petite". Plus exactement, nous considérons la famille d'équations :

$$\begin{cases} \mathrm{d}v_t^\varepsilon = -\mathrm{sgn}(v_t^\varepsilon)b(v_t^\varepsilon)\mathrm{d}t + \varepsilon\mathrm{d}\ell_t, \\ \mathrm{d}x_t^\varepsilon = v_t^\varepsilon\mathrm{d}t \end{cases}$$

où ℓ est un processus de Lévy symétrique α-stable. On peut voir ci-dessous deux simulations des solutions v^ε et v de l'équation précédente et de l'équation déterministe.

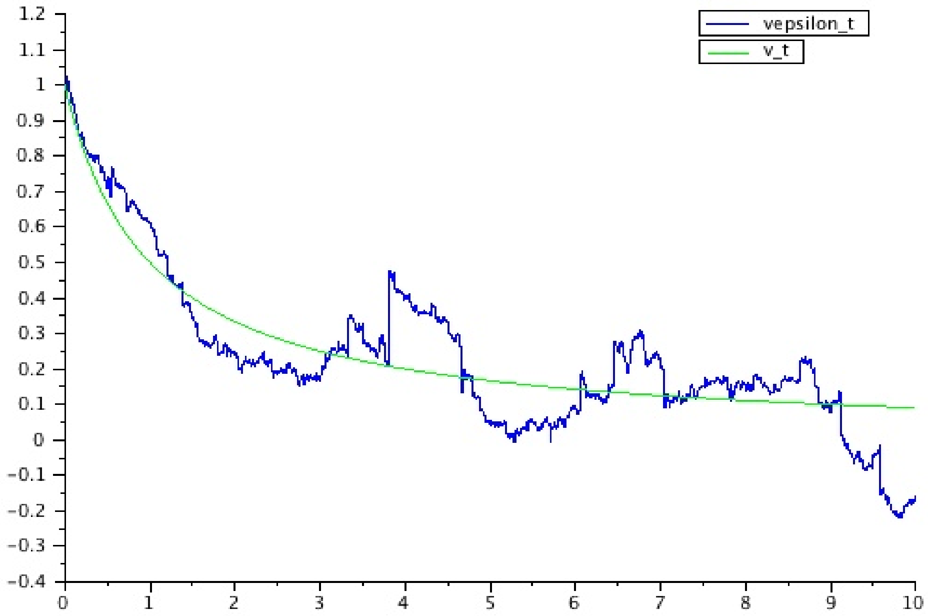

Figure 1: Perturbation par un processus de Lévy symétrique 1.5-stable de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = v^2$.

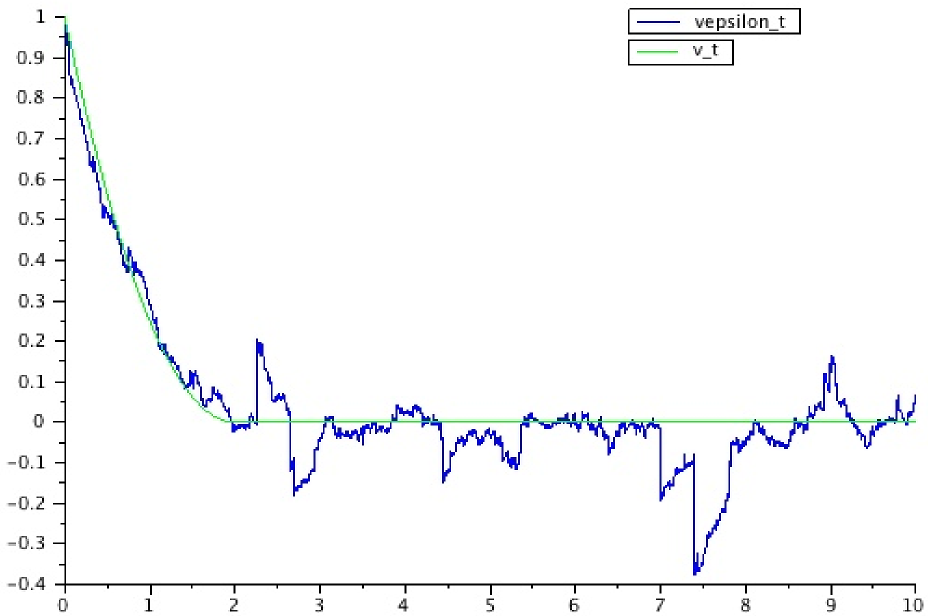

Figure 2: Perturbation par un processus de Lévy symétrique 1.5-stable de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = \sqrt{|v|}$.

La première question que l'on peut se poser dans ce cas est : la famille de solutions converge-t-elle vers la solution de l'équation déterministe ? Si oui, peut-on donner une vitesse de convergence ou des comportements extrêmes ? Dans le cas brownien, cette question a été énormement étudiée. L'étude de convergence, de sortie d'un compact, de grandes déviations ou encore de développement limité font l'objet de nombreux articles. Sur ce dernier point, qui est celui que nous étudierons, on peut citer les travaux de N. Ikeda et S. Watanabe (cf. [21] p. 532).

Dans le cas où la perturbation est un processus de Lévy, on peut citer les travaux de P. Imkeller et I. Pavlyukevich [20] qui étudient également le temps de sortie de familles de solutions d'équations en les comparant à la solution déterministe. Ils démontrent le résultat suivant :

Théorème (Imkeller-Pavlyukevich)**.** *Soit X^ε la solution de l'équation différentielle stochastique*

$$X_t^\varepsilon = x - \int_0^t U'(X_{s^-}^\varepsilon)\mathrm{d}s + \varepsilon \ell_t,$$

où $x \in \mathbb{R}$ et ℓ est un processus de Lévy, somme d'un mouvement brownien standard et d'un processus de Lévy α-stable indépendant avec $\alpha \in (0,2)$. On suppose que U est $\mathcal{C}^3$ dans un voisinage suffisamment grand de 0 et vérifie $U(0) = 0$, $U'(x) = 0$ ssi $x = 0$, $\forall x \in \mathbb{R}$, $U'(x)x \geq 0$ et $U''(0) > 0$. De plus, pour garantir l'existence et l'unicité de la solution, on suppose que U' est localement lipschitzienne et croît à l'infini plus vite que linéairement. Enfin, pour $a > 0$ et $b > 0$, on introduit $\sigma(\varepsilon) := \inf\{t \geq 0 : X_t^\varepsilon \notin [-b, a]\}$. Il existe des constantes positives ε_0, γ, δ et $C > 0$ telles que, pour $0 < \varepsilon < \varepsilon_0$, on a

$$\exp\left(-u\varepsilon^\alpha \frac{\theta}{\alpha}(1 + C\varepsilon^\delta)\right)(1 - C\varepsilon^\delta) \leq P_x(\sigma(\varepsilon) > u) \leq \exp\left(-u\varepsilon^\alpha \frac{\theta}{\alpha}(1 - C\varepsilon^\delta)\right)(1 + C\varepsilon^\delta)$$

uniformément pour tout $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$ et $u \geq 0$ avec $\theta = \frac{1}{a^\alpha} + \frac{1}{b^\alpha}$.

La preuve de ce résultat s'appuie sur le lemme suivant :

Lemme. *Soit $T \geq 0$, $c > 0$ et $\gamma = \frac{2-\alpha}{5}$. Alors, sous les hypothèses du théorème, il existe ε_0 et C tels que pour tous $0 < \varepsilon \leq \varepsilon_0$ et $x \in [-b, a]$, on a*

$$P_x\left(\sup_{[0,T]} |X_t^\varepsilon - x_t| \geq c\varepsilon^\gamma\right) \leq CT\varepsilon^{\alpha + \gamma/2}$$

où x est la solution de l'équation déterministe.

Remarquons que dans ces résultats, la régularité demandée à la force de frottement est toujours strictement supérieure à l'ordre du développement que l'on souhaite obtenir. Dans le Chapitre 2, nous donnons le développement de la solution autour de la solution déterministe mais nous utilisons la structure attractive des forces de frottements pour avoir une regularité du même ordre que celle du développement. Notons enfin qu'étudier un processus de Lévy quelconque pose plusieurs difficultés. Dans les Chapitres 2 et 3, nous nous restreignons aux processus α-stables symétriques et nous essayons d'enlever l'hypothèse de symétrie dans le Chapitre 4. Donnons le résultat principal du deuxième chapitre :

Proposition. *On considère la famille d'équations différentielles stochastiques*

$$\begin{cases} dv_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - sgn(v_t^\varepsilon)|v_t^\varepsilon|^\beta \mathrm{d}t, \quad v_0^\varepsilon = v_0 \neq 0 \\ x_t^\varepsilon = \int_0^t v_s^\varepsilon \mathrm{d}s, \end{cases}$$

où ℓ est un processus de Lévy symétrique α-stable.

- *Pour $\beta > 0$, quand $\varepsilon \to 0$, $\{v_t^\varepsilon : t \geq 0\}$ (respectivement x^ε) converge vers v (respectivement x) en probabilité uniformement sur tout intervalle compact (UCP).*
- *Pour $\beta \geq 1$, soit Z la solution de l'eds $Z_t = -\int_0^t \mathcal{U}''(v_s))Z_s \mathrm{d}s + \ell_t$. On a $\frac{1}{\varepsilon}(v^\varepsilon - v - \varepsilon Z)$ et $\frac{1}{\varepsilon}(x^\varepsilon - x - \varepsilon \int_0^\cdot Z_s \mathrm{d}s)$ convergent UCP vers 0, quand $\varepsilon \to 0$.*

Remarquons que dans le cas $\alpha = 2$, le premier résultat peut être étendu à $\beta = 0$ mais pour $\alpha < 2$ et $\beta = 0$, on ne sait pas s'il y a existence d'une solution car la dérive n'est plus continue.

La preuve du premier point est un peu différente suivant si $\alpha = 2$ ou $\alpha < 2$. Dans le cas brownien, on applique la formule d'Itō à $|v_t^\varepsilon - v_t|^p$ puis les inégalités BDG et de Gronwall qui, grâce au caractère attractif de la dérive, permettent de montrer la convergence de v^ε vers v en norme L^p. Dans le cas $\alpha < 2$, on ne peut plus effectuer directement les mêmes étapes. On va donc utiliser un procédé d'entrelacement pour s'y ramener. On considère en effet Y^ε, solution de la même équation que v^ε mais où on a retiré les "grands" sauts du processus de Lévy. On peut alors utiliser la même méthode que précédemment pour montrer que $\mathbb{E}|Y_t^\varepsilon - v_t|^p$ converge vers 0. Il reste ensuite à montrer que la contribution des "grands" sauts est négligeable en montrant que v^ε vers v en norme L^p pour $p < \alpha$.

Pour le deuxième point, la preuve est la même dans les deux cas et repose sur deux idées. On démontre tout d'abord que Z se comporte comme ℓ i.e. :

$$\sup_{s\in[0,T]} |Z_s| \leq C_{\beta,T,v_0} \sup_{s\in[0,T]} |\ell_s|$$

et on en déduit que εZ est bien une partie du développement au premier ordre. Puis il faut montrer que le reste $R_t := v_t^\varepsilon - v_t - \varepsilon Z_t$ est bien négligeable devant ε. On démontre en effet que, sur l'événement $\left\{ \sup\limits_{s\in[0,T]} |\varepsilon \ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}} \right\}$, on a presque sûrement

$$\sup_{s\in[0,T]} |R_s| \leq C''_{\beta,T,v_0} \Big(\sup_{s\in[0,T]} |\varepsilon \ell_s| \Big)^2.$$

Le résultat est alors prouvé en considérant la dernière inégalité et la probabilité du complémentaire de l'événement $\left\{ \sup\limits_{s\in[0,T]} |\varepsilon \ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}} \right\}$.

La proposition précédente entraîne donc que la solution perturbée va rester proche de la solution déterministe lorsque la vitesse initiale n'est pas nulle en raison du caractère attractif de l'équation. Il est donc naturel d'étudier, dans un second temps, le cas où la vitesse initiale est nulle. La solution déterministe est alors la fonction nulle et on se demande si la solution va rester proche de 0. Plus précisément, 0 est un point d'équilibre stable et attractif. La question que l'on va se poser dans le troisième chapitre est : comment la perturbation aléatoire va-t-elle influencer le mouvement de la particule en temps long si la vitesse initiale est nulle ? En effet, l'accumulation de petites perturbations pourrait entraîner une déviation de la vitesse et de la position de la particule par rapport à la vitesse et la position déterministes. Plus exactement, si on introduit, pour $t \geq 0$,

$$\mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad \mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon,$$

quels sont les comportements de $\mathcal{X}^\varepsilon$ et de $\mathcal{V}^\varepsilon$ quand ε tend vers 0 ? Dans le cas de frottement linéaire ($b(v) = |v|$), R. Hintze et I. Pavlyukevich ont montré dans [19] que $\mathcal{X}^\varepsilon$ converge en loi vers un processus de Lévy symétrique α-stable en utilisant la structure linéaire et la connaissance de la solution exacte de cette équation. Mais qu'en est-il dans les cas où b n'est plus linéaire, la solution n'ayant plus de forme explicite ? Donnons quelques simulations pour représenter ce problème.

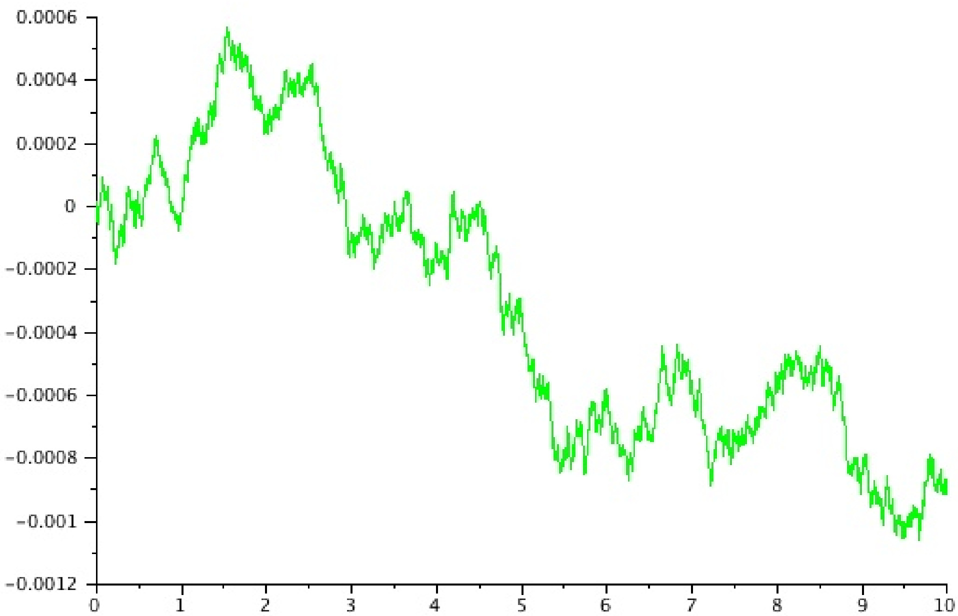

Figure 3: $\mathcal{X}^\varepsilon$: perturbation en temps long par un processus brownien de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = \mathbb{1}_{\mathbb{R}\setminus\{0\}}(v)$.

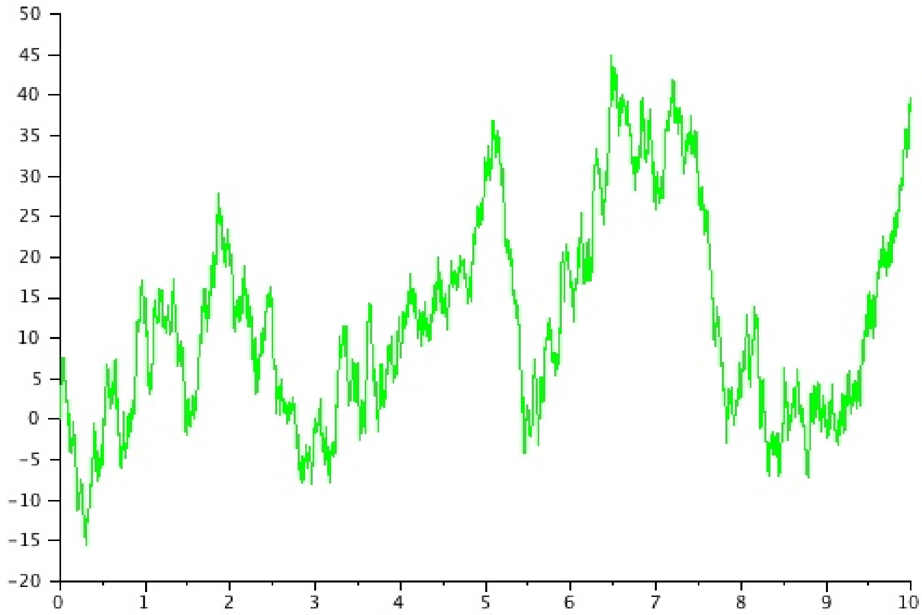

Figure 4: $\mathcal{X}^\varepsilon$: perturbation en temps long par un processus brownien de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = v^2$.

En comparant ces deux graphiques avec le résultat de R. Hintze et I. Pavlyukevich, on remarque qu'il semble y avoir convergence de la position vers un mouvement brownien mais avec des changements d'échelle en espace. On montrera en effet dans la première partie du Chapitre 3 :

Théorème. *Dans le cas gaussien, $\alpha = 2$, et $\beta > -1$, si on pose $\theta = \frac{2}{\beta+1}$, alors il existe une constante strictement positive $\kappa_{2,\beta}$ telle que le processus*

$$\{\varepsilon^{(\beta-1)\theta} x^\varepsilon_{\varepsilon^{-2}t} : t \geq 0\} = \{\varepsilon^{(\beta-1)\theta} \mathcal{X}^\varepsilon_t : t \geq 0\}$$

converge en distribution, lorsque $\varepsilon \to 0$, dans l'espace des fonctions continues $\mathrm{C}([0,\infty))$ *muni de la topologie uniforme sur les compacts, vers un mouvement brownien de variance $\kappa_{2,\beta}$.*

On a donc un coefficient d'agrandissement ou de rétrécissement selon si $\beta \leq 1$ ou non. On peut alors se demander si ce résultat se généralise au cas $\alpha < 2$.

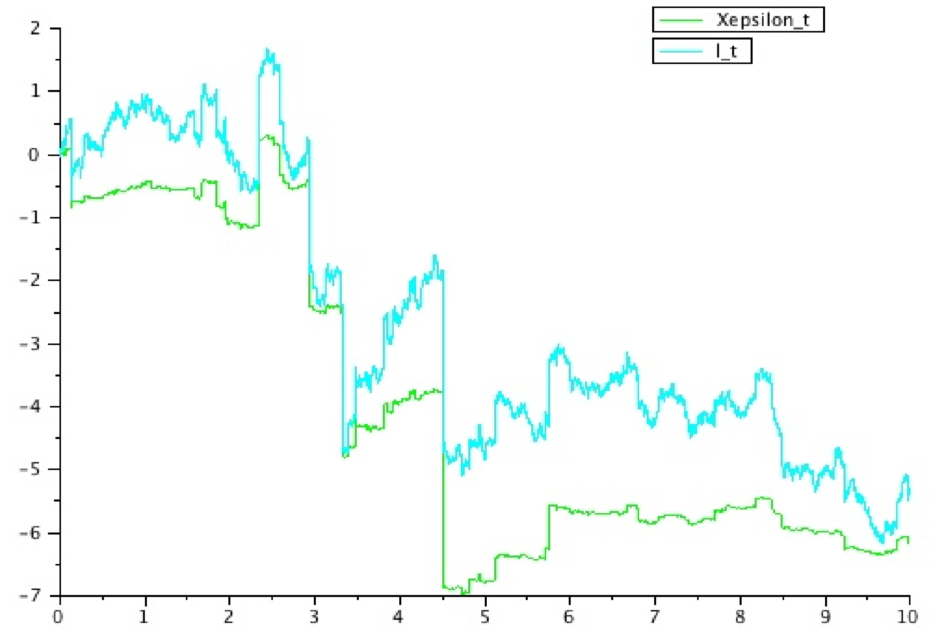

Figure 5: $\mathcal{X}^\varepsilon$: perturbation en temps long par un processus de Lévy symétrique 1.5-stable de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = \mathbb{1}_{\mathbb{R}\setminus\{0\}}(v)$.

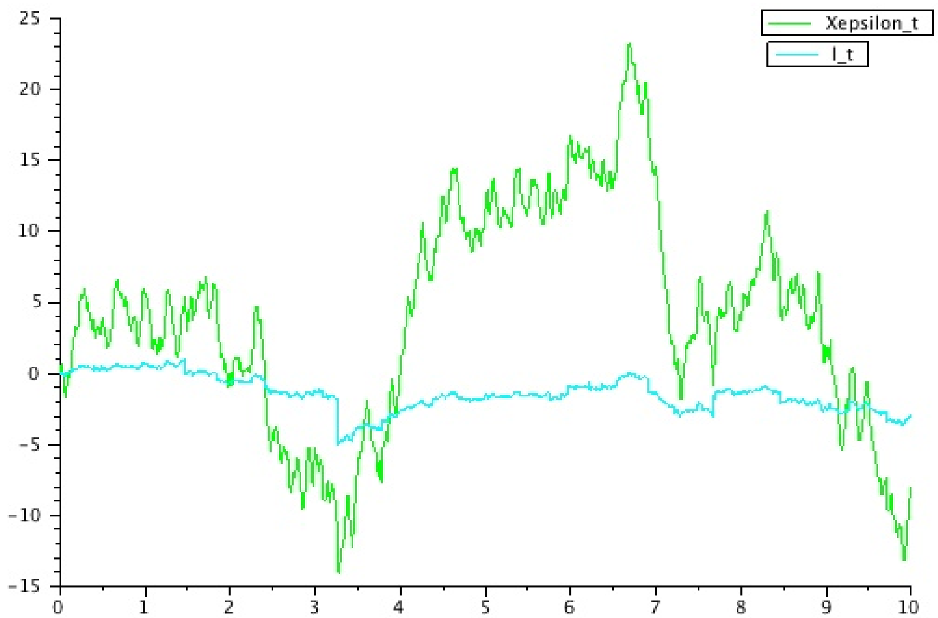

Figure 6: $\mathcal{X}^\varepsilon$: perturbation en temps long par un processus de Lévy symétrique 1.5-stable de coefficient $\varepsilon = 0.01$ et de force de frottement $b(v) = v^2$.

Sur le premier graphique, on observe une convergence vers un processus stable, ce qui est cohérent avec le résultat de R. Hintze et I. Pavlyukevich mais dans le deuxième cas, il semble que les sauts disparaissent et que l'ordre de grandeur ne soit plus le même, ce qui serait alors plutôt cohérent avec le cas brownien où $\beta > 1$. On montrera en effet dans la deuxième partie du Chapitre 3 :

Théorème. *Dans le cas $\alpha \in (0, 2)$, supposons que $\beta + \frac{\alpha}{2} > 2$ et posons $\theta = \frac{\alpha}{\alpha+\beta-1}$. Alors il existe une constante strictement positive $\kappa_{\alpha,\beta}$ telle que le processus*

$$\{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta} x^{\varepsilon}_{\varepsilon^{-\alpha}t} : t \geq 0\} = \{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta} \mathcal{X}^{\varepsilon}_t : t \geq 0\}$$

converge en distribution, lorsque $\varepsilon \to 0$, dans l'espace des fonctions continues $\mathrm{C}([0, \infty))$ *muni de la topologie uniforme sur les compacts, vers un mouvement brownien de variance $\kappa_{\alpha,\beta}$.*

La preuve dans les cas $\alpha = 2$ et $\alpha < 2$ est la même, néanmoins, dans le cas $\alpha = 2$, la plupart des calculs sont explicites grâce aux fonctions d'échelle et aux mesures vitesse. Elle s'articule en 6 étapes.

1) Dans un premier temps, on pose

$$L^{\varepsilon}_t := \frac{\ell_{t\,\varepsilon^{-(\beta-1)\theta}}}{\varepsilon^{(\beta-1)\theta/\alpha}} \quad \text{et} \quad V^{\varepsilon}_t := \frac{\mathcal{V}^{\varepsilon}_{t\,\varepsilon^{\alpha\theta}}}{\varepsilon^{\theta}},$$

où $\theta = \frac{\alpha}{\alpha+\beta-1}$. Par auto-similarité, L^{ε} a la même loi qu'un processus de Lévy α-stable et on a

$$\mathcal{X}^{\varepsilon}_t = \varepsilon^{(2-\beta)\theta} \int_0^{t\varepsilon^{-\alpha\theta}} V^{\varepsilon}_s \mathrm{d}s \quad \text{et} \quad V^{\varepsilon}_t = L^{\varepsilon}_t - \int_0^t \mathrm{sgn}(V^{\varepsilon}_s)|V^{\varepsilon}_s|^{\beta} \mathrm{d}s.$$

Comme ℓ est supposé symétrique, V^{ε} l'est également donc $\mathbb{E}V^{\varepsilon}_t = 0$. Comme V^{ε} ne dépend plus de ε en loi, on peut relier notre problème concernant l'étude du comportement asymptotique de x^{ε} à un théorème ergodique d'ordre deux. En effet, dans le cas $\alpha = 2$, les propriétés sur les fonctions d'échelle et les mesures vitesse permettent de montrer que V^{ε} est ergodique. Dans le cas $\alpha < 2$, on utilise le théorème de Kulik (voir [25]) :

Théorème (Kulik)**.** *Soit Y la solution de $\mathrm{d}Y_t = -f(Y_t)\mathrm{d}t + \ell_t$. On suppose que f est localement lipschitzienne et que $\lim\sup_{|y|\to\infty} \frac{f(y)}{y} > 0$. On suppose de plus que ν, la mesure de Lévy de ℓ, satisfait les conditions suivantes :*

1. *il existe $q > 0$: $\int_{|u|>1} |u|^q \nu(\mathrm{d}u) < +\infty$,*
2. *$\nu(\mathbb{R} \setminus \{0\}) \neq 0$.*

Alors Y est exponentiellement ergodique, i.e. sa distribution invariante m existe et est unique, et pour une certaine constante positive C, pour tout point de départ $x \in \mathbb{R}$, si on note P_t la loi de Y_t,

$$||P_t - m||_{var} = \underset{t\to\infty}{O}(\exp[-Ct]).$$

2) On va démontrer la convergence de $\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}\mathcal{X}^\varepsilon$ vers un mouvement brownien en utilisant le théorème de Whitt (voir [45]) :

Théorème (Whitt)**.** *Soient $n \geq 1$ et $M_n := (M_{n,1}, \ldots, M_{n,k})$ une martingale locale dans $\mathcal{D}^k$ satisfaisant $M_n(0) = (0, \ldots, 0)$. Soit $C = (c_{i,j})$ une matrice de covariance de taille $k \times k$, i.e. une matrice symétrique définie positive. On définit enfin pour un processus à sauts x, $J(x,T) = \sup_{0\leq t\leq T} |x(t) - x(t-)|$.*

On suppose que M_n est localement de carré intégrable et que l'on peut donc définir le processus de covariation quadratique prévisible $\langle M_{n,i}, M_{n,j}\rangle$. On suppose également que les trois propriétés suivantes sont satisfaites :

1. $\lim_{n\to+\infty} \mathbb{E}[J(\langle M_{n,i}, M_{n,j}\rangle, T)] = 0$ *pour tous i, j et T,*
2. $\lim_{n\to+\infty} \mathbb{E}[J(M_n^2, T)] = 0$ *pour tout T,*
3. $\lim_{n\to+\infty} \langle M_{n,i}, M_{n,j}\rangle(t) = c_{i,j}t$ *pour tous i et j.*

Alors M_n converge en distribution vers un mouvement brownien centré de covariance C.

Pour utiliser ce théorème, il faut décomposer $\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}\mathcal{X}_t^\varepsilon$ sous la forme d'une martingale locale et d'un terme de reste.

3) Soit $\mathcal{A}$ le générateur infinitésimal associé à l'eds

$$\mathrm{d}V_t^\varepsilon = -\mathrm{sgn}(V_t^\varepsilon)|V_t^\varepsilon|^\beta \mathrm{d}t + \mathrm{d}L_t.$$

On veut résoudre l'équation de Poisson $\mathcal{A}g = Id$ car la formule d'Itō appliquée à g et V^ε donne

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{\alpha\theta}{2}}\Big[g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big] - \varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}.$$

où M est une martingale locale. Dans le cas $\alpha = 2$, l'équation de Poisson est une équation différentielle d'ordre deux dont on connaît une solution explicite. Dans le cas $\alpha < 2$, on a une équation intégro-différentielle que l'on ne sait pas résoudre directement. On utilise alors le théorème de Glynn et Meyn (voir [17]). En effet, si on trouve une fonction de Liapounov h satisfaisant de bonnes conditions, l'équation de Poisson admet alors une solution g qui vérifie en plus $g \leq c(h+1)$.

4) On démontre alors que $\varepsilon^{\frac{\alpha\theta}{2}}\Big[g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big]$ converge vers 0 uniformément sur tout compact, en utilisant l'expression de g dans le cas $\alpha = 2$ et en utilisant la majoration $g \leq c(h+1)$ et le fait que h soit une fonction de Liapounov dans le cas $\alpha < 2$.

5) À l'aide du théorème de Whitt, on démontre que $-\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}$ converge vers un mouvement brownien. Dans le cas $\alpha = 2$, il n'y a pas de saut, il suffit de vérifier la convergence de $\langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\cdot\,\varepsilon^{-\alpha\theta}} \rangle(t)$. Comme on connaît la mesure invariante de V^ε grâce à la mesure vitesse, on peut vérifier directement que le théorème ergodique s'applique et ensuite conclure. Dans le cas $\alpha < 2$, l'idée est la même mais on n'a plus de forme exacte pour la mesure invariante. On démontre tout d'abord que les deux premiers points du théorème de Whitt sont vérifiés en utilisant le fait que h est une fonction de Liapounov (comme dans 4)). On démontre ensuite que le théorème ergodique s'applique, sous la condition $\beta + \frac{\alpha}{2} > 2$, en utilisant la forme de h et l'estimation de la queue de la mesure invariante donnée par le théorème de Samorodnitsky et Grigoriu (voir [38]) :

   Théorème (Samorodnitsky, Grigoriu)**.** *On considère Y, la solution de l'eds $Y_t = Y_0 - \int_0^t f(Y_s)\mathrm{d}s + L_t$. On suppose que la fonction f est localement lipschitzienne, que $f(0) = 0$, que f est croissante sur $\mathbb{R}$, qu'elle varie régulièrement à l'infini avec un exposant $\beta > 1$ i.e. for all $a > 0$, $\lim\limits_{x\to+\infty} \frac{f(ax)}{f(x)} = a^\beta$. De plus, pour certaines constantes $A > 0$ et β_1, pour tout $x \leq -1$, $-f(-x) \geq Ax^{\beta_1}$. On suppose que ν, la mesure de Lévy de L, est symétrique et vérifie $\nu([u, +\infty)) = u^{-\alpha} l(u)$ où l est une fonction à variation lente à l'infini i.e. for all $a > 0$, $\lim\limits_{x\to+\infty} \frac{l(ax)}{l(x)} = 1$. Alors l'unique mesure invariante m de Y satisfait*

   $$\lim_{u\to\infty} \frac{m([u,\infty))}{h(u)} = 1,$$

   avec $h(u) = \int_u^\infty \frac{\nu([y,\infty))}{f(y)} \mathrm{d}y$.

6) On peut conclure en utilisant le théorème de Slutsky et le "continuous mapping theorem". Notons que dans le cas $\alpha < 2$, comme on travaille dans l'espace de Skorokhod, l'addition n'est pas continue mais le résultat s'applique car la limite est continue et la convergence a donc lieu dans l'espace des fonctions continues.

On peut résumer les résultats obtenus ou connus sous la forme du graphique suivant. Notons que le comportement de la position pour $\alpha < 2$, $\beta + \frac{\alpha}{2} < 2$ et $\beta \neq 1$ est une question ouverte et fait l'objet de travaux en cours.

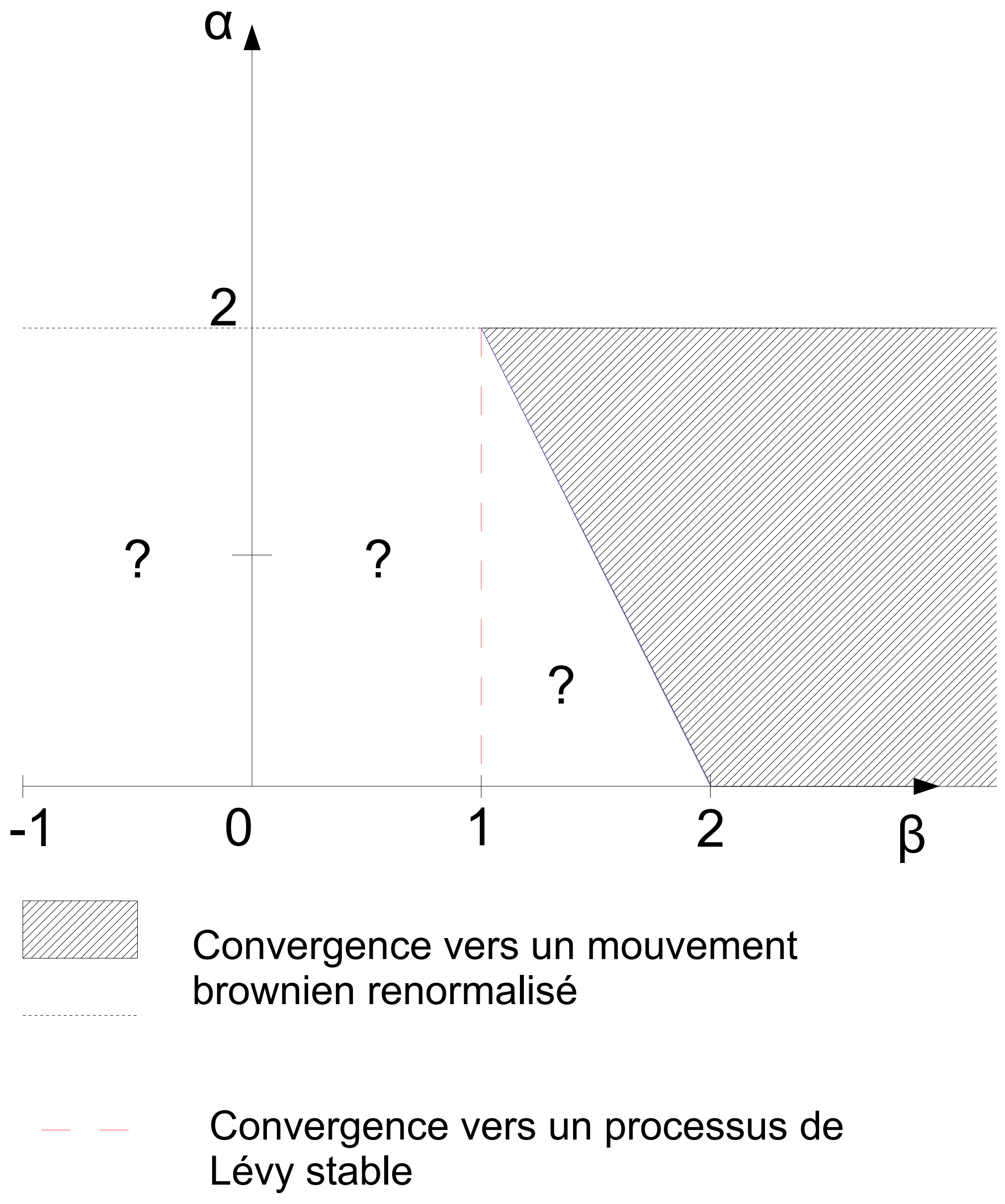

Figure 7: Graphique récapitulatif des résultats.

Dans le Chapitre 4, nous verrons comment étendre les résultats du Chapitre 3 lorsque la perturbation aléatoire n'est plus symétrique. On démontre en effet, avec les mêmes notations que dans le troisième chapitre :

Théorème. *On suppose que $\beta + \frac{\alpha}{2} > 2$. Alors il existe une constante positive $\kappa_{\alpha,\beta}$ telle que le processus*

$$\left\{\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\left(x^{\varepsilon}_{\varepsilon^{-\alpha}t} - \varepsilon^{\theta-\alpha}t\int x m_{\alpha,\beta}(\mathrm{d}x)\right) : t \geq 0\right\}$$

converge en distribution vers un mouvement brownien avec coefficient de diffusion $\kappa_{\alpha,\beta}$, quand $\varepsilon \to 0$. $m_{\alpha,\beta}$ est la distribution invariante de V.

La preuve est exactement la même que dans le cas symétrique, en tenant compte de la dérive dans le générateur infinitésimal de V et en changeant l'équation de Poisson en $\mathcal{A}g = \mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x)$. Néanmoins, le résultat de Samorodnitsy et Grigoriu nécessite que le processus de Lévy soit symétrique. Nous avons étendu, dans un certain sens, ce résultat pour le cas non symétrique. On démontre le théorème suivant :

Théorème. *On considère Y solution de l'eds $Y_t = Y_0 - \int_0^t f(Y_s)\mathrm{d}s + L_t$. On suppose que la fonction f est localement lipschitzienne, que $f(0) = 0$, que f est croissante sur $\mathbb{R}$, qu'elle varie régulièrement à l'infini avec un exposant $\beta > 1$ i.e. for all $a > 0$, $\lim\limits_{x\to+\infty} \frac{f(ax)}{f(x)} = a^{\beta}$. De plus, pour certaines constantes $A > 0$ et β_1, pour tout $x \leq -1$, $-f(-x) \geq Ax^{\beta_1}$. On suppose que L est un processus de Lévy α-stable non symétrique (mais pas asymétrique). On note, pour tout $u > 0$*

$$h(u) := \int_u^{+\infty} \frac{\nu((y,+\infty))}{f(y)}\mathrm{d}y\,.$$

Alors

$$\lim_{u\to+\infty} \frac{\mathbb{P}_x(Y_t > u)}{h(u)} = 1,$$

uniformément en $x \in \mathbb{R}$ et $t \geq t_0$.

On déduit le résultat suivant :

Corollaire. *Sous les mêmes hypothèses que dans le théorème précédent, on a*

$$\lim_{u\to+\infty} \frac{m_{\alpha,\beta}((u,+\infty))}{h(u)} = 1$$

La preuve de ce théorème repose sur l'idée suivante : sur un intervalle de temps fixé bien choisi, Y a plus de chance de dépasser la valeur u grâce à une accumulation de "petits" sauts et un "grand" saut plutôt que par une accumulation seule de "petits" sauts ou par l'occurrence de deux "grands" sauts. La preuve est donc séparée en 4 parties :

1) on découpe d'abord astucieusement les "très petits" sauts pour se ramener à un processus centré ayant un nombre fini de sauts ;

2) on utilise la structure de l'équation pour démontrer que l'accumulation de "petits" sauts sur un intervalle de temps bien choisi ne peut pas faire dépasser un certain seuil dépendant de u ;

3) on démontre ensuite que la contribution des termes ne correspondant à aucun "grand" saut ou à plus de deux "grands" sauts va être négligeable ;

4) on estime enfin la contribution du terme dominant pour faire apparaître le résultat.

Enfin, dans le Chapitre 5, on utilise les résultats des chapitres précédents pour étudier le cas où la perturbation est la somme d'un mouvement brownien et d'un processus de Lévy α-stable. On considère

$$\begin{cases} v_t^1 = -\int_0^t \mathrm{sgn}(\mathrm{v_s^1})|\mathrm{v_s^1}|^\beta \mathrm{ds} + \varepsilon \mathrm{b_t} + \varepsilon \ell_\mathrm{t}, \quad \mathrm{v_0^1} = 0 \\ x_t^1 = \int_0^t v_s^1 \mathrm{d}s, \end{cases}$$

où ℓ est un processus de Lévy α-stable avec $\alpha \in (0,2)$ (symétrique ou non mais pas asymétrique) et b est un mouvement brownien standard. Les normalisations obtenues dans les Chapitres 3 et 4 montrent que le bruit dominant va être le processus de Lévy. On va comparer cette solution avec celle du cas où le bruit est seulement un processus de Lévy α-stable. On démontre, qu'avec les mêmes notations que dans les Chapitres 3 et 4, $\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}\Big(X_t^1 - X_t^2\Big)$ converge uniformément sur tout compact vers 0 où X^1 est associé à la solution x_1 et X^2 à la solution de l'eds avec uniquement un processus de Lévy. La preuve repose sur le fait que $X^1 - X^2$ est continue et sur les majorations classiques obtenues avec la formule d'Itō et le lemme de Gronwall.

Dans une dernière partie, nous commencerons à étudier le cas de la dimension deux avec des dérives disjointes et des bruits browniens corrélés. On démontrera dans ce cas, que l'on peut se ramener au cas de la dimension un pour toutes les estimations et utiliser le théorème de Whitt bi-dimensionnel.

Perspectives

Plusieurs directions d'étude pourront être envisagées pour la suite. Dans un premier temps, dans le Chapitre 3, nous avons obtenus des résultats pour $\alpha = 2$ et pour $\alpha < 2$ et $\beta + \frac{\alpha}{2} > 2$. On pourrait étudier le comportement du processus position pour $\beta + \frac{\alpha}{2} \leq 2$ et $\alpha < 2$. Les simulations numériques ainsi que les résultats obtenus pour $\beta = 1$ permettent de conjecturer, dans le cas $\beta + \frac{\alpha}{2} < 2$ et $\alpha < 2$, une convergence de $x_{\varepsilon^{-\alpha} t}$ vers un processus de Lévy stable dont l'indice pourrait être $\frac{\alpha}{2-\beta}$. Le cas de la droite $\beta + \frac{\alpha}{2} = 2$ ressemble à une transition de phase. En effet, la conjecture précédente pour la droite, donne une convergence vers un processus de Lévy 2-stable donc un mouvement brownien. Le résultat que l'on a obtenu dans le Chapitre 3 donnerait, sur la droite, la convergence du processus position vers un mouvement brownien avec coefficient de normalisation 1 donc le même processus. L'intégrabilité d'ordre $(\beta + \frac{\alpha}{2})^-$ du processus position pourrait être à l'origine de cette transition.

Plusieurs autres problèmes peuvent être considérés dans le cadre de la dimension un. Tout d'abord, le résultat du Chapitre 3 a été démontré dans le cas où la force de frottement est une puissance (entière ou non) de la vitesse. Toutefois, tous les résultats intermédiaires ne semblent utiliser que le comportement asymptotique de la force de frottement. Peut- on alors étendre le résultat dans le cas où la force de frottement est supposée juste équivalente à une puissance de v à l'infini ? Une première piste de réponse à cette question est que le comportement de la force de frottement au voisinage de l'infini va conditionner le retour en 0 et donc la régularisation des grands sauts du processus de Lévy. Cependant, la régularisation des "petits" sauts qui donne pour limite un mouvement brownien est conditionnée par le comportement au voisinage de 0. La question est donc ouverte.

Dans une autre direction, le premier pas de la preuve du résultat du Chapitre 3 repose sur l'autosimilarité du processus de Lévy. Peut-on généraliser les résultats en considérant un mouvement brownien fractionnaire qui est lui aussi autosimilaire ? Dans ce cas, la disparition du caractère markovien risque d'être problématique mais certaines pistes utilisant la théorie des chemins rugueux peuvent être envisagées.

La troisième question que l'on peut se poser est : que se passe-t-il lorsqu'on ajoute une fonction diffusive devant le processus de Lévy ? Cette question semble très difficile car il y a peu de résultats concernant l'ergodicité dans ce cas.

Enfin, la dernière piste d'étude consiste à changer le type de forces. Que se passe-t-il si on considère une force répulsive ? Différents articles sur le phénomène de Péano (voir [3], [12], [13]) traitent de ce cas de figure. Le cas où la force reste attractive mais ne possède plus un mais deux points fixes stables peut également se poser. La simulation numérique semble suggérer que lorsque la perturbation tend à disparaître, la vitesse va tendre vers un des deux points fixes de façon aléatoire. Dans l'article [10], ce cas de figure est discuté.

Dans un second temps, l'étude de la dimension deux n'est pas complète. En effet, l'équation de frottement classique est

$$\begin{cases} \mathrm{d}v_t = -\frac{v_t}{||v_t||}||v_t||^{\beta}\mathrm{d}t + \mathrm{d}\ell_t, \\ \mathrm{d}x_t = v_t\mathrm{d}t, \end{cases}$$

où ℓ est un processus dont les deux composantes sont des processus de Lévy indépendants et on peut se demander s'il est possible d'étendre les résultats obtenus dans les chapitres précédents. On peut noter que, dans ce cas, les propriétés d'ergodicité et celles liées à la fonction de Liapounov restent vraies mais il faudrait généraliser le résultat de Samorodnitsky et Grigoriu dans le cas de la dimension 2.

Pour finir, la question de la dimension supérieure à deux se pose puisqu'elle est le cadre privilégié de la physique. Néanmoins, ne serait-ce que dans le cas brownien, le fait que le mouvement brownien tri-dimensionnel ne soit pas récurrent risque de poser de nombreux problèmes.

Chapter 1

SDEs driven by Lévy processes

One can find all the results on the Lévy processes, given in this chapter, in [40] and [1], and the results on the Brownian motion in [36], [22] and [37].

1.1 Generalities on Lévy processes

1.1.1 Definition and properties of Lévy processes

In his article [27], Paul Lévy introduces stochastic processes that generalize the Brownian motion, considering not only continuous processes but also processes with jumps. He defines the Lévy process :

Definition 1.1.1. *We consider a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, a time* $T \in \mathbb{R}_+$ *and a process* $\{L_t,\, 0 \leq t \leq T\}$ *whith takes values in* $\mathbb{R}$*. We say that* L *is a Lévy process if*

1. *Its increments are independent : for any* $0 \leq t_1 \leq \cdots \leq t_n \leq T$*, the family* $(L_{t_{i+1}} - L_{t_i})_{1 \leq i \leq n-1}$ *is independent.*
2. *Its increments are stationary : for any* $0 \leq s \leq t \leq T$*,* $L_t - L_s$ *has the same distribution as* $L_{t-s} - L_0$*.*
3. L *is continuous in probability : for any* $0 \leq t \leq T$*,* $\lim_{h \to 0} \mathbb{P}(|L_{t+h} - L_t| > \varepsilon) = 0$*, for all* $\varepsilon > 0$*.*

Often, one can find a definition where we replace the last item by "The process is càdlàg" (right continuous with limit on the left) but the two definitions are equivalent. Let see some examples.

1. The Brownian motion is a Lévy process.
2. The simple Poisson process is a Lévy process.
3. The compound Poisson process is also a Lévy process.

More generally, we can see that, in a certain sense, we have only these processes.

Theorem 1.1.2 (Lévy-Itô's decomposition)**.** *For a Lévy process L, we define a random measure on compact set A by $N(t,A) := \#\{0 \le s \le t,\, L_s - L_{s^-} \in A\}$, for all $t \in [0,T]$. For all A, the process $N(\cdot, A)$ is a Poisson process with intensity $\nu(A) := \mathbb{E}(N(1,A))$ called the Lévy measure. Denote $\tilde{N}(t,A) = N(t,A) - t\nu(A)$ its compensated. Then, there exist $b \in \mathbb{R}$, $\sigma \ge 0$ and a standard Brownian motion B such that*

$$L_t = L_0 + bt + \sigma B_t + \int_{|x|\le 1} x\tilde{N}(t, \mathrm{d}x) + \int_{|x|>1} xN(t, \mathrm{d}x).$$

This means that all Lévy processes are sum of a drift, a Brownian motion and a Poisson process.

1.1.2 Markov's property

One of the most important properties of Lévy processes is that a Lévy process is a strong Markov process. Introduce the family of operators $(P_t)_{t\in[0,T]}$ on $L^\infty(\mathbb{R})$ defined for all $f \in L^\infty(\mathbb{R})$ by $P_t f(x) = \mathbb{E}_x[f(L_t)]$. $(P_t)_{t\in[0,T]}$ is the Markov semigroup associated to the process L. It verifies the properties :

- $P_0 = Id$
- for all t and s, $P_t \circ P_s = P_{t+s}$
- for all t, $||f||_\infty \le 1 \Rightarrow ||P_t f|| \le 1$.

Lévy processes are even Feller processes. Define $C_0(\mathbb{R})$ the subset of $L^\infty(\mathbb{R})$ of the continuous function which have 0 for limit at infinity.

Proposition 1.1.3 (Feller's property)**.** *Assume that L is a Lévy process, and denote (P_t) its semigroup then*

- *for all t, $P_t C_0(\mathbb{R}) \subset C_0(\mathbb{R})$*
- *for all $f \in C_0(\mathbb{R})$, $\lim_{t\to 0} ||P_t f - f||_\infty = 0$.*

We can also define the infinitesimal generator, $\mathcal{L}$, of L and using the Lévy-Itô decomposition, one can write: for all $g \in \mathcal{C}^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$

$$\mathcal{L}g(x) = bg'(x) + \frac{\sigma^2}{2} g''(x) + \int_{\mathbb{R}} \left[g(x+y) - g(x) - yg'(x)\mathbb{1}_{|y|\le 1} \right] \nu(\mathrm{d}y)$$

1.1.3 Stable Lévy processes

We will study a particular class of Lévy processes: the α-stable processes.

Definition 1.1.4.

1. *Let μ be an infinitely divisible probability measure on $\mathbb{R}$. It is called strictly stable if, for any $a > 0$, there is $b > 0$ such that $\hat{\mu}(z)^a = \hat{\mu}(bz)$.*

2. *Let L be a Lévy process, it is called strictly stable if the distribution of L_1 is strictly stable.*

This definition is closely related to the notion of self-similarity.

Definition 1.1.5. *Let X be a stochastic process on $\mathbb{R}$. It is called self-similar if, for any $a > 0$, there is $b > 0$ such that $\{X_{at} : t \geq 0\} \stackrel{d}{=} \{bX_t : t \geq 0\}$.*

Theorem 1.1.6. *Let L be a Lévy process, L is strictly stable if and only if L is self-similar. In this case, there exists $\alpha \in (0, 2]$ such that $b = a^{\frac{1}{\alpha}}$ and L is called α-stable process.*

Stable processes are interesting for two reasons. First, due to the self-similarity, we will be able to study a process at different scaling. Secondly, the Lévy measure and the Lévy-Itô decomposition is almost totally explicit for the one dimensional case.

Theorem 1.1.7. *Let L be an α-stable process.*

1. *If $\alpha = 2$ then L is a Brownian motion (not necessarily standard), so without jump.*

2. *If $\alpha < 2$ then L is a pure jump process (so without a Brownian part) with Lévy measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}[a_+\mathbb{1}_{z>0} + a_-\mathbb{1}_{z<0}]\mathrm{d}z$ and Lévy-Itô decomposition:*
   - $L_t = \int_0^t \int_{\mathbb{R}^*} xN(t, \mathrm{d}x)$ *if* $\alpha \in (0, 1)$,
   - $L_t = \int_0^t \int_{\mathbb{R}^*} x\tilde{N}(t, \mathrm{d}x)$ *if* $\alpha \in (1, 2)$,
   - $L_t = bt + \int_0^t \int_{|z|\leq 1} x\tilde{N}(t, \mathrm{d}x) + \int_0^t \int_{|z|>1} xN(t, \mathrm{d}x)$ *if* $\alpha = 1$.

Remark 1.1.8. *If $\alpha < 2$, we can write $L_t = bt + \int_0^t \int_{|z|\leq 1} x\tilde{N}(t, \mathrm{d}x) + \int_0^t \int_{|z|>1} xN(t, \mathrm{d}x)$ but if $\alpha \neq 1$, b is unique whereas if $\alpha = 1$, b can take any value in $\mathbb{R}$.*

Finally, we need a supplementary notion.

Definition 1.1.9. *Let X be a one-dimensional stochastic process. X is called symmetric if the distributions of X and $-X$ are the same.*

Proposition 1.1.10.

1. *If $\alpha = 2$ then an α-stable Lévy process is always symmetric.*

2. *If $\alpha = 1$ then the Lévy measure of an α-stable Lévy process is always symmetric ($a_+ = a_-$).*

3. *Let $\alpha < 2$ and L be a symmetric α-stable Lévy process, then $L_0 = 0$, $a_+ = a_-$ and $b = 0$ in the latter theorem.*

We now have all what we need to start the study of stochastic differential equations driven by α-stable Lévy processes. Let us notice that there is a big difference between the case $\alpha = 2$ with a continuous process and $\alpha < 2$ with pure jumps processes so we will always distinguish both cases in the statement of results.

1.2 Stochastic calculus and differential equations driven by α-stable Lévy processes

1.2.1 Stochastic calculus

Assume that L is a Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that it takes its values in $\mathbb{R}$. Denote ν its Lévy measure. We define the Lévy-type stochastic integral.

Definition 1.2.1.

1. *We define $\mathcal{P}_2(\nu)$ to be the set of all equivalence classes of mappings $F : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ which coincide almost everywhere with respect to $\nu \times \mathbb{P}$ and which are predictable and satisfy $\mathbb{P}\left(\int_0^T \int_{\mathbb{R}} |F(t,x)|^2 \nu(\mathrm{d}x)\mathrm{d}t < \infty\right) = 1$.*

2. *We define $\mathcal{P}_2$ to be the set of all equivalence classes of mappings $F : [0,T] \times \Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\mathbb{P}$ and which are predictable and satisfy $\mathbb{P}\left(\int_0^T |F(t)|^2 \mathrm{d}t < \infty\right) = 1$.*

3. *We say that a process Y is a Lévy-type stochastic integral if it can be written in the form:*

$$Y_t = Y_0 + \int_0^t G(s)\mathrm{d}s + \int_0^t F(s)\mathrm{d}B_s + \int_0^t \int_{|x|\leq 1} H(s,x)\tilde{N}(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_{|x|>1} K(s,x)N(\mathrm{d}s,\mathrm{d}x),$$

with, for all T, $|G|^{\frac{1}{2}} \in \mathcal{P}_2$, $F \in \mathcal{P}_2$, $H\mathbb{1}_{|x|\leq 1} \in \mathcal{P}_2(\nu)$ and K only predictable.

For this kind of integral that generalizes Itô's integral, we can also extend the Itô's formula.

Theorem 1.2.2 (Itô-Lévy's formula)**.** *Assume that Y is a Lévy-type stochastic integral and that $f \in \mathcal{C}^2$ then we have, for all t,*

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t f'(Y_{s^-})G(s)\mathrm{d}s + \int_0^t f'(Y_{s^-})F(s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(Y_{s^-})F^2(s)\mathrm{d}s \\ &+ \int_0^t \int_{|x|\leq 1} [f(Y_{s^-} + H(s,x)) - f(Y_{s^-})]\tilde{N}(\mathrm{d}s,\mathrm{d}x) \\ &+ \int_0^t \int_{|x|\leq 1} [f(Y_{s^-} + H(s,x)) - f(Y_{s^-}) - f'(Y_{s^-})H(s,x)]\nu(\mathrm{d}x)\mathrm{d}s \\ &+ \int_0^t \int_{|x|>1} [f(Y_{s^-} + K(s,x)) - f(X_{s^-})]N(\mathrm{d}s,\mathrm{d}x) \end{aligned}$$

The last important definition is the quadratic variation.

Definition 1.2.3. *Let Y be a Lévy-type stochastic integral, we define the quadratic variation of Y by*

$$[Y]_t = \int_0^t F^2(s)\mathrm{d}s + \int_0^t \int_{|x|\le 1} H^2(s,x)N(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_{|x|>1} K^2(s,x)N(\mathrm{d}s,\mathrm{d}x).$$

Proposition 1.2.4. *There is a direct relation with the Meyer angle bracket*

$$\begin{aligned}\langle Y\rangle_t &= [Y]_t - \textstyle\int_0^t \int_{|x|\le 1}(H^2(s,x)\mathbb{1}_{|x|\le 1} + K^2(s,x)\mathbb{1}_{|x|>1})\tilde{N}(\mathrm{d}s,\mathrm{d}x)\\ &= \textstyle\int_0^t F^2(s)\mathrm{d}s + \int_0^t \int (H^2(s,x)\mathbb{1}_{|x|\le 1} + K^2(s,x)\mathbb{1}_{|x|>1})\nu(\mathrm{d}x)\mathrm{d}s.\end{aligned}$$

1.2.2 Equations driven by α-stable Lévy processes

In this section, we will consider a Lévy-type stochastic integral Y solution of the following equation

$$\begin{aligned}Y_t = Y_0 + \int_0^t b(Y_{s^-})\mathrm{d}s + \int_0^t \sigma(Y_{s^-})\mathrm{d}B_s\\ + \int_0^t \int_{|x|\le 1} H(Y_{s^-},x)\tilde{N}(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_{|x|>1} K(Y_{s^-},x)N(\mathrm{d}s,\mathrm{d}x).\end{aligned}$$

To simplify the notation, we will always forget that the time in the equation are s^- and we will only note s.

We introduce two conditions

(GLC) Global Lipschitz condition There exists $K_1 > 0$ such that, for all y_1, y_2,

$$|b(y_1)-b(y_2)|^2+|\sigma(y_1)-\sigma(y_2)|^2+\int_{|x|\le 1}|H(y_1,x)-H(y_2,x)|^2\nu(\mathrm{d}x) \le K_1|y_1-y_2|^2.$$

(GGC) Global growth condition There exists K_2 such that, for all y,

$$|b(y)|^2 + |\sigma(y)|^2 + \int_{|x|\le 1}|H(y,x)|^2\nu(\mathrm{d}x) \le K_2(1+|y|^2).$$

Theorem 1.2.5. *Assume **(GLC)** and **(GGC)**. Assume that, for all $x > 1$, the function $y \mapsto K(y,x)$ is continuous. Then there exists a unique solution to the later stochastic differential equation.*

We note that the condition on K is really weak but **(GLC)** and **(GGC)** are strong and not always fulfilled. We have a local version of those conditions and theorem.

Definition 1.2.6. *Let T_∞ be a stopping time and suppose that $Y = (Y_t, 0 \le t < T_\infty)$ is a solution of the latter equation. If $T_\infty < \infty$ a.s., we say that Y is a local solution and T_∞ is called the explosion time for the SDE.*

(LLC) Local Lipschitz condition For all $n \in \mathbb{N}$, there exists $K_1(n) > 0$ such that, for all y_1, y_2 with $\max(|y_1|, |y_2|) \leq n$,

$$|b(y_1)-b(y_2)|^2+|\sigma(y_1)-\sigma(y_2)|^2+\int_{|x|\leq 1} |H(y_1,x)-H(y_2,x)|^2\nu(\mathrm{d}x) \leq K_1(n)|y_1-y_2|^2.$$

(LGC) Local growth condition For all $n \in \mathbb{N}$, there exists $K_2(n)$ such that, for all y with $|y| < n$,

$$|b(y)|^2 + |\sigma(y)|^2 + \int_{|x|\leq 1} |H(y,x)|^2\nu(\mathrm{d}x) \leq K_2(n)(1+|y|^2).$$

Theorem 1.2.7. *Assume* **(LLC)** *and* **(LGC)**. *Assume that, for all $x > 1$, the function $y \mapsto K(y,x)$ is continuous. Then there exists a unique local solution Y to the stochastic differential equation defined up to an explosion random time T_∞. Moreover, we have $|Y_{T_\infty^-}| = \infty$.*

In the Brownian case ($H = K = 0$), we can define the scale function and the speed measure to extend results.

Definition 1.2.8. *Assume that σ is strictly positive and that $b\sigma^{-2}$ is locally integrable then we define the scale function and the speed measure by*

$$s(x) := \int_0^x \exp\left(-2\int_0^y b(z)\sigma(z)^{-2}\mathrm{d}z\right)\mathrm{d}y \quad \textit{and} \quad m(\mathrm{d}x) := [\sigma(x)^2 s'(x)]^{-1}\mathrm{d}x\,. \tag{1.2.1}$$

Theorem 1.2.9. *Assume that σ is strictly positive and that $b\sigma^{-2}$ is locally integrable. Assume that $\int_0^\infty s'(x)m([0,x])\mathrm{d}x = \infty$. Then there exists a unique global solution to the stochastic differential equation without jump.*

It is not possible to define the scale function and the speed measure for a general Lévy-type stochastic integral. To show the non explosion, we can use Lyapunov's function. In the case that we are studying, we give another result to obtain the non explosion.

1.2.3 A special case

The proposition of this section is based on the interlacing procedure used in [15]. We consider the stochastic differential equation

$$Y_t = Y_0 - \int_0^t f(Y_s)\mathrm{d}s + L_t,$$

where L is an α-stable Lévy process with $\alpha < 2$.

Proposition 1.2.10. *Assume that f is locally Lipschitz, $f(0) = 0$, f is nondecreasing and $\lim_{y\to\pm\infty} f(y) = \pm\infty$. Then there exists a unique global solution Y to the stochastic differential equation. Moreover, we have, for all $\delta \in (0,\alpha)$ and all $T > 0$,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^\delta\Big] < \infty.$$

Proof. Using Theorem 1.2.7, there exists a unique local solution Y up to an explosion time τ. Then we just have to show that, for all $\delta \in (0,\alpha)$ and all $T>0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^\delta\Big] < \infty.$$

By the Lévy-Itô's decomposition, there exist a Poisson process N, its compensated $\tilde{N}$ and a constant b such that

$$L_t = bt + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z).$$

So the equation satisfied by Y, starting from a point x, is

$$Y_t = x + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) - \int_0^t f(Y_s)\mathrm{d}s + bt. \quad (1.2.2)$$

Firstly, we skip the big jumps term and show that the resulting process X has moments of any order and, secondly, we use an interlacing procedure to handle the process Y. In fact, we consider the equation

$$X_t = x + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \int_0^t f(X_s)\mathrm{d}s + bt, \quad (1.2.3)$$

and apply Itô-Lévy's formula with $x\mapsto x^2$. We obtain

$$\begin{aligned} X_t^2 &= x^2 + \tilde{M}_t + \int_0^t \int_{|z|\le 1} [(X_s+z)^2 - X_s^2 - 2zX_s]\nu(\mathrm{d}z)\mathrm{d}s \\ &\qquad - 2\int_0^t X_s f(X_s)\mathrm{d}s + 2b\int_0^t X_s\mathrm{d}s \\ &= x^2 + \tilde{M}_t + t\int_{|z|\le 1} z^2\nu(\mathrm{d}z) - 2\int_0^t X_s f(X_s)\mathrm{d}s + 2b\int_0^t X_s\mathrm{d}s, \quad (1.2.4) \end{aligned}$$

where the local martingale term is given by

$$\tilde{M}_t := \int_0^t \int_{|z|\le 1} [(X_s+z)^2 - X_s^2]\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

The constants depending only on α (respectively on α and b) will be denoted c_α (respectively $k_{\alpha,b}$) and could change from line to line in this proof. Let us write the third term in (1.2.4) as $c_\alpha t$, $\lim_{|y|\to\infty}(c_\alpha - 2yf(y) + 2by) = -\infty$ so

$$X_t^2 \le x^2 + k_{\alpha,b}t + \tilde{M}_t. \quad (1.2.5)$$

By Kunita's inequality (see for instance [1], p. 265) and by our convention on constants,

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le t} X_s^2\right] &\le x^2 + k_{\alpha,b}t + c_\alpha \int_0^t \int_{|z|\le 1} \mathbb{E}\left[(X_s+z)^2 - X_s^2\right]^2 \nu(\mathrm{d}z)\mathrm{d}s \\ &\le x^2 + k_{\alpha,b}t + c_\alpha\int_0^t \mathbb{E}[X_s^2]\mathrm{d}s \le x^2 + k_{\alpha,b}t + c_\alpha\int_0^t \mathbb{E}\left[\sup_{0\le u\le s} X_u^2\right]\mathrm{d}s. \quad (1.2.6) \end{aligned}$$

Applying Gronwall's inequality, we get

$$\mathbb{E}\left[\sup_{0\le u\le t} X_u^2\right] \le (x^2 + k_{\alpha,b}t)e^{c_\alpha t}. \tag{1.2.7}$$

Hence $\tilde{M}$ is a (true) square integrable martingale and, taking expectation in (1.2.5), we obtain

$$\mathbb{E}[X_t^2] \le x^2 + k_{\alpha,b}t. \tag{1.2.8}$$

Re-injecting this in (1.2.6), we get that, for any $T > 0$, there exists a positive constant $C_{\alpha,b,T}$ depending also on T, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} X_t^2\Big] \le C_{\alpha,b,T}(1+x^2). \tag{1.2.9}$$

We proceed with the study of (1.2.2). Denote by $0 = T_0 < T_1 < T_2 < \dots$ the jumping times of N restricted to $\{|z| > 1\}$, and by (Z_n) the jumps which are i.i.d. random variables with distribution $\lambda^{-1}\mathbb{1}_{\{|z|>1\}}\nu(\mathrm{d}z)$, where $\lambda := \int_{\{|z|>1\}} \nu(\mathrm{d}z)$. Therefore $\int_0^t \int_{|z|>1} zN(\mathrm{d}s, \mathrm{d}z) = \sum_{n\in\mathbb{N}} Z_n \mathbb{1}_{\{T_n\le t\}}$ and (1.2.2) coincides with (1.2.3) on each time interval (T_n, T_{n+1}). Since Y is a solution of (1.2.3) on $[0, T_1)$, by using (1.2.9),

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} Y_t^2 \Big| \mathcal{G}\Big] \le C_{\alpha,b,T}(1+x^2) \text{ a.s.,} \qquad \text{with } \mathcal{G} := \sigma(T_1, T_2, \dots).$$

By using the Jensen inequality and the classical inequality $(|a|+|b|)^\delta \le c_\delta(|a|^\delta+|b|^\delta)$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} |Y_t|^\delta \Big| \mathcal{G}\Big] \le C_{\alpha,b,\delta,T}(1+|x|^\delta) \text{ a.s.}$$

Moreover, $Y_{T_1} = Y_{T_1-} + Z_1$, hence $|Y_{T_1}|^\delta \le c_\delta\big(|Y_{T_1-}|^\delta + |Z_1|^\delta\big)$. Since $\delta < \alpha$, we have $\mathbb{E}(|Z_1|^\delta) < \infty$ and consequently,

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T]} |Y_t|^\delta \Big| \mathcal{G}\Big] \le C_{\alpha,b,\delta,T}(1+|x|^\delta) \text{ a.s.}$$

By using the strong Markov property and the latter inequality on (T_n, T_{n+1}), but starting from Y_{T_n}, we can show that, for any $n \ge 0$,

$$u_n := \mathbb{E}\Big[\sup_{t\in[T_n\wedge T, T_{n+1}\wedge T]} |Y_t|^\delta \Big| \mathcal{G}\Big] \le C'_{T,\delta}(1 + \mathbb{E}[|Y_{T_n}|^\delta|\mathcal{G}]).$$

Then the sequence $(u_n)_{n\ge0}$ satisfies $u_0 \le C'_{T,\delta}$ and $u_{n+1} \le C'_{T,\delta}(1+u_n)$, implying that there exists $C_{T,\delta} > 1$ such that $u_n \le C_{T,\delta}^{n+1}$. We deduce that

$$\mathbb{E}\Big[\sup_{t\in[0,T_n\wedge T]} |Y_t|^\delta \Big| \mathcal{G}\Big] \le u_0 + \dots + u_{n-1} \le \frac{C_{T,\delta}^{n+1}}{C_{T,\delta}-1} \text{ a.s.}$$

Finally,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^\delta\Big] &\le \sum_{n\ge0} \mathbb{E}\Big[\mathbb{1}_{T_n<T<T_{n+1}} \mathbb{E}\big(\sup_{t\in[0,T_n\wedge T]} Y_t^\delta|\mathcal{G}\big)\Big] \\ &\le \frac{1}{C_{T,\delta}-1}\sum_{n\ge0} C_{T,\delta}^{n+2}\frac{(\lambda T)^n}{n!}e^{-\lambda T} < \infty.\end{aligned}$$

□

1.3 Ergodicity and invariant measure

We still consider Y solution of the stochastic differential equation

$$Y_t = Y_0 + \int_0^t b(Y_s)\mathrm{d}s + \int_0^t \sigma(Y_s)\mathrm{d}B_s \\ + \int_0^t \int_{|x|\le 1} H(Y_s, x)\tilde{N}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x|>1} K(Y_s, x)N(\mathrm{d}s, \mathrm{d}x).$$

We assume that we have a unique solution to this equation. Since a Lévy process is a Markov process, we can show, using the uniqueness of the solution, that Y is also a Markov process. We are interested in the ergodicity of the process Y.

In the Brownian case, we can still use the scale function and the speed measure.

Proposition 1.3.1. *If $s(\infty) = \infty$ and $m(\mathbb{R}) < \infty$, the diffusion Y is regular and is a recurrent and ergodic process with the invariant distribution $\frac{1}{m(\mathbb{R})}m$. Therefore, for all $f \in \mathrm{L}^1(m)$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y_s)\mathrm{d}s = \frac{1}{m(\mathbb{R})}\int_{\mathbb{R}} f(x)m(\mathrm{d}x), \quad \textit{almost surely.} \tag{1.3.10}$$

For general processes, the classical argument uses Lyapunov function but in the case that we are looking for, there is an easier way to obtain the result. We consider Y, the solution of

$$Y_t = Y_0 - \int_0^t f(Y_s)\mathrm{d}s + L_t,$$

where L is a pure jump Lévy process.

Proposition 1.3.2 (Kulik)**.** *Assume that f is locally Lipschitz and $\limsup_{|y|\to\infty} \frac{f(y)}{y} > 0$. Assume that ν satisfies the following conditions:*

1. *there exists $q > 0$: $\int_{|u|>1} |u|^q \nu(\mathrm{d}u) < +\infty$,*
2. *$\nu(\mathbb{R} \setminus \{0\}) \neq 0$.*

Then Y is exponentially ergodic, i.e. its invariant distribution m exists and is unique and for some positive constant C, for all starting point $x \in \mathbb{R}$, if we denote by P_t the distribution of Y_t,

$$||P_t - m||_{var} = \underset{t\to\infty}{O}(\exp[-Ct]).$$

In the Brownian case, the invariant distribution is explicit and coincides with the speed measure. But in the general Lévy case, the invariant distribution is not explicit. However, in certain case, the asymptotic tail of the measure is known.

Theorem 1.3.3 (Samorodnitsky, Grigoriu)**.** *We consider Y solution of the differential equation $Y_t = Y_0 - \int_0^t f(Y_s)\mathrm{d}s + L_t$. We assume that f is locally Lipschitz, $f(0) = 0$, f is nondecreasing on $\mathbb{R}$, f is regularly varying at infinity with exponent $\beta > 1$ and for some constants $A > 0$ and β_1, for all $x \le -1$, $-f(-x) \ge Ax^{\beta_1}$.*

Assume that ν is symmetric and satisfies $\nu([u,+\infty)) = u^{-\alpha}l(u)$ where l is slowly varying at infinity. Then the unique invariant distribution m of Y satisfies

$$\lim_{u\to\infty} \frac{m([u,\infty))}{h(u)} = 1,$$

with $h(u) = \int_u^\infty \frac{\nu([y,\infty))}{f(y)}\mathrm{d}y$.

We will see in Chapter 4 how we can obtain the same result without the assumption that the Lévy process is symmetric.

Chapter 2

Stability of a differential equation with small symmetric α-stable perturbations

In this chapter, we will study the stochastic differential equation

$$\begin{cases} dv_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - \frac{1}{2}\mathcal{U}'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = v_0 \neq 0 \\ x_t^\varepsilon = \int_0^t v_s^\varepsilon \mathrm{d}s, \end{cases}$$

where the potential is $\mathcal{U}(x) = \frac{2}{\beta+1}|x|^{\beta+1}$ with $\beta > 0$ and ℓ is a symmetric α-stable Lévy process. Thanks to Theorem 1.2.7, Theorem 1.2.9 for $\alpha = 2$ and Proposition 1.2.10 and Theorem 170 in [41] for $\alpha < 2$, we know that this equation has a unique global solution. Moreover, in the case $\alpha = 2$, there exists a unique solution for $\beta = 0$. The potential is attractive so we can prove that, when ε will go to 0, the solutions v^ε and x^ε will converge toward the solutions of the deterministic differential equation

$$\begin{cases} dv_t = -\frac{1}{2}\mathcal{U}'(v_t)\mathrm{d}t, \quad v_0^\varepsilon = v_0 \\ x_t = \int_0^t v_s \mathrm{d}s. \end{cases}$$

We can wonder what is the convergence's rate of the solution v^ε (respectively x^ε) toward v (respectively x). The proof follows the idea of [19] by doing a Taylor expansion and using the attractive structure of the equation to assume less regularity on $\mathcal{U}$.

2.1 Convergence of the speed and of the position toward the solutions of the deterministic equation

We start by proving the convergence of v^ε toward v and of x^ε toward x. We split the proof into two parts since the cases $\alpha = 2$ and $\alpha < 2$ behave differently.

Proposition 2.1.1. *For $\beta > 0$, as $\varepsilon \to 0$, $\{v_t^\varepsilon : t \geq 0\}$ (respectively $\{x^\varepsilon : t \geq 0\}$) converges toward v (respectively x) in probability uniformly on each compact interval.*

Remark 2.1.2. *In the Brownian case $\alpha = 2$, we can generalize this result for $\beta = 0$.*

Proof. We begin with the Brownian case. By Itô's formula, we have

$$|v_t^\varepsilon - v_t|^2 = 2\int_0^t (v_s^\varepsilon - v_s)\mathrm{d}(v_s^\varepsilon - v_s) + \langle v_t^\varepsilon - v_t\rangle$$
$$= \int_0^t (v_s^\varepsilon - v_s)(-\mathcal{U}'(v_s^\varepsilon) + \mathcal{U}'(v_s))\mathrm{d}s + 2\varepsilon\int_0^t (v_s^\varepsilon - v_s)\mathrm{d}b_s + \varepsilon^2 t.$$

Since $\mathcal{U}'$ is increasing, $(v_s^\varepsilon - v_s)(-\mathcal{U}'(v_s^\varepsilon) + \mathcal{U}'(v_s)) \leq 0$ so

$$|v_t^\varepsilon - v_t|^2 \leq 2\varepsilon\int_0^t (v_s^\varepsilon - v_s)\mathrm{d}b_s + \varepsilon^2 t. \tag{2.1.1}$$

Fix $T > 0$. Since $(a+b)^2 \leq 2(a^2+b^2)$ and $|x|^2 \leq 1 + |x|^4$, by using the Burkholder-Davis-Gundy inequality, we can see that, for all $t \leq T$, there exists a positive constant C_2 such that

$$\mathbb{E}\Big[\sup_{0\leq u\leq t} |v_u^\varepsilon - v_u|^2\Big]^2 \leq 2\varepsilon^4 T^2 + 8\varepsilon^2\mathbb{E}\Big[\sup_{0\leq u\leq t}\int_0^u (v_s^\varepsilon - v_s)\mathrm{d}b_s\Big]^2$$
$$\leq 2\varepsilon^4 T^2 + 8\varepsilon^2 C_2\int_0^t \mathbb{E}\Big[|v_s^\varepsilon - v_s|^2\Big]\mathrm{d}s$$
$$\leq 2\varepsilon^4 T^2 + 8\varepsilon^2 C_2 T + 8\varepsilon^2 C_2\int_0^t \mathbb{E}\Big[\sup_{0\leq u\leq s}|v_u^\varepsilon - v_u|^4\Big]\mathrm{d}s. \tag{2.1.2}$$

So, if $t \in [0,T]$, using Gronwall's lemma, we get

$$\mathbb{E}\Big[\sup_{0\leq u\leq t}|v_u^\varepsilon - v_u|^4\Big] < \infty,$$

so $\int_0^t (v_s^\varepsilon - v_s)\mathrm{d}b_s$ is a true square integrable martingale. By taking expectation in (2.1.1), we get

$$\mathbb{E}|v_t^\varepsilon - v_t|^2 \leq \varepsilon^2 t. \tag{2.1.3}$$

Injecting (2.1.3) in the second inequality of (2.1.2), we deduce the statement for v^ε :

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|v_t^\varepsilon - v_t|^4\Big] \leq 2\varepsilon^4 T^2 + 4\varepsilon^4 C_2 T^2. \tag{2.1.4}$$

Since

$$x_t^\varepsilon - x_t = \int_0^t (v_s^\varepsilon - v_s)\mathrm{d}s,$$

the proof is complete for the case $\alpha = 2$, by using Jensen's inequality and (2.1.4).

In the case $\alpha < 2$, we need to use an interlacing procedure as in the proof of Proposition 1.2.10. By Itô-Lévy's decomposition, there exists a Poisson measure N with intensity $\nu(\mathrm{d}z)\mathrm{d}s = |z|^{-1-\alpha}\mathrm{d}z\mathrm{d}s$, and its compensated $\tilde{N}$ such that

$$\ell_t = \int_0^t\int_{|z|\leq 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z).$$

The equation satisfied by v^ε, assuming that v^ε is starting from $x+v_0 \in \mathbb{R}$ arbitrary, can be written

$$v_t^\varepsilon = x+v_0+\varepsilon \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z)+\varepsilon \int_0^t \int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z)-\frac{1}{2}\int_0^t \mathcal{U}'(v_s^\varepsilon)\mathrm{d}s. \quad (2.1.5)$$

Consider another equation where we skip the third big jumps term

$$Y_t^\varepsilon = x+v_0+\varepsilon \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \frac{1}{2}\int_0^t \mathcal{U}'(Y_s^\varepsilon)\mathrm{d}s, \quad (2.1.6)$$

and apply Itô-Lévy's formula. We obtain, since $\mathcal{U}'$ is increasing,

$$|Y_t^\varepsilon - v_t|^2 = x^2+\tilde{M}_t^\varepsilon + \int_0^t \int_{|z|\le 1} [(Y_s^\varepsilon - v_s+\varepsilon z)^2-(Y_s^\varepsilon - v_s)^2 - 2\varepsilon z(Y_s^\varepsilon - v_s)]\nu(\mathrm{d}z)\mathrm{d}s$$
$$+\int_0^t (Y_s^\varepsilon - v_s)(-\mathcal{U}'(Y_s^\varepsilon)+\mathcal{U}'(v_s))\mathrm{d}s \le x^2+\tilde{M}_t^\varepsilon+\varepsilon^2 t\int_{|z|\le 1} z^2\nu(\mathrm{d}z), \quad (2.1.7)$$

where the local martingale term is given by

$$\tilde{M}_t^\varepsilon := \int_0^t \int_{|z|\le 1} [(Y_s - v_s+\varepsilon z)^2 - (Y_s - v_s)^2]\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

In this proof, the constants depending only on α will be denoted c_α or c'_α and could change from line to line. There exists a positive constant c_α such that, for all $t \ge 0$,

$$|Y_t^\varepsilon - v_t|^2 \le x^2+\varepsilon^2 c_\alpha t+\tilde{M}_t^\varepsilon. \quad (2.1.8)$$

By Kunita's inequality (see for instance [1], p. 265),

$$\mathbb{E}\left[\sup_{0\le s\le t}(Y_s^\varepsilon - v_s)^2\right] \le x^2+\varepsilon^2 c_\alpha t+c'_\alpha \int_0^t \int_{|z|\le 1} \mathbb{E}\left[(Y_s^\varepsilon - v_s+\varepsilon z)^2-(Y_s^\varepsilon - v_s)^2\right]^2\nu(\mathrm{d}z)\mathrm{d}s$$
$$\le x^2+\varepsilon^2 c_\alpha t+c'_\alpha \int_0^t \int_{|z|\le 1} \mathbb{E}\left[2\varepsilon^4 z^4+8\varepsilon^2 z^2(Y_s^\varepsilon - v_s)^2)\right]\nu(\mathrm{d}z)\mathrm{d}s$$
$$\le x^2+(\varepsilon^2+\varepsilon^4)c_\alpha t+\varepsilon^2 c'_\alpha \int_0^t \mathbb{E}[(Y_s^\varepsilon - v_s)^2]\mathrm{d}s$$
$$\le x^2+(\varepsilon^2+\varepsilon^4)c_\alpha t+\varepsilon^2 c'_\alpha \int_0^t \mathbb{E}\left[\sup_{0\le u\le s}(Y_u^\varepsilon - v_u)^2\right]\mathrm{d}s. \quad (2.1.9)$$

Applying Gronwall's inequality, we get

$$\mathbb{E}\left[\sup_{0\le u\le t}(Y_u^\varepsilon - v_u)^2\right] \le (x^2+(\varepsilon^2+\varepsilon^4)c_\alpha t)e^{\varepsilon^2 c'_\alpha t}.$$

Hence $\tilde{M}^\varepsilon$ is a (true) square integrable martingale and, taking expectation in (2.1.8), we obtain

$$\mathbb{E}[Y_t^\varepsilon - v_t]^2 \le x^2+\varepsilon c_\alpha t.$$

Re-injecting this in (2.1.9), we get that, for any $T > 0$, there exists a positive constant $C_{\alpha,T}$ depending also on T, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}(Y_t^\varepsilon - v_t)^2\Big] \le C_{\alpha,T}(\varepsilon^2 + x^2). \tag{2.1.10}$$

We proceed with the study of (2.1.5). Denote by $0 = T_0 < T_1 < T_2 < \dots$ the jumping times of N restricted on $\{|z| > 1\}$, and by (Z_n) the hight of the jumps which are i.i.d. random variables with distribution $\lambda^{-1}\mathbb{1}_{\{|z|>1\}}\nu(\mathrm{d}z)$, where $\lambda := \int_{\{|z|>1\}}\nu(\mathrm{d}z)$. Therefore

$$\int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) = \sum_{n\in\mathbb{N}} Z_n\mathbb{1}_{\{T_n\le t\}}$$

and (2.1.5) coincides with (2.1.6) on each time interval (T_n, T_{n+1}). Since v^ε is a solution of (2.1.6) on $[0, T_1)$, by using (2.1.10),

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)}(v_t^\varepsilon - v_t)^2\Big|\mathcal{G}\Big] \le C_{\alpha,T}(\varepsilon^2 + x^2), \qquad \text{with } \mathcal{G} := \sigma(T_1, T_2, \dots).$$

Fix $\delta < \alpha$, by Jensen inequality and using the classical inequality $(a+b)^\delta \le c_\delta(a^\delta + b^\delta)$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)}|v_t^\varepsilon - v_t|^\delta\Big|\mathcal{G}\Big] \le C_{\alpha,\delta,T}(\varepsilon^\delta + |x|^\delta).$$

Furthermore, $v_{T_1}^\varepsilon - v_{T_1} = v_{T_1-}^\varepsilon - v_{T_1-} + \varepsilon Z_1$, hence $|v_{T_1}^\varepsilon - v_{T_1}|^\delta \le c_\delta\big(|v_{T_1-}^\varepsilon - v_{T_1-}|^\delta + \varepsilon^\delta|Z_1|^\delta\big)$. Since $\delta < \alpha$, $\mathbb{E}(|Z_1|^\delta) < \infty$. Consequently, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T]}|v_t^\varepsilon - v_t|^\delta\Big|\mathcal{G}\Big] \le C_{\alpha,\delta,T}(\varepsilon^\delta + |x|^\delta).$$

Using the same inequality on (T_n, T_{n+1}), with starting point $v_{T_n}^\varepsilon - v_{T_n}$, we can show that, for any $n \ge 0$,

$$u_n := \mathbb{E}\Big[\sup_{t\in[T_n\wedge T, T_{n+1}\wedge T]}|v_t^\varepsilon - v_t|^\delta\Big|\mathcal{G}\Big] \le C'_{T,\delta}(\varepsilon^\delta + \mathbb{E}[|v_{T_n}^\varepsilon - v_{T_n}|^\delta|\mathcal{G}]).$$

Then the sequence $(u_n)_{n\ge0}$ satisfies $u_0 \le C'_{T,\delta}\varepsilon^\delta$ and $u_{n+1} \le C'_{T,\delta}(\varepsilon^\delta + u_n)$, implying that there exists $C_{T,\delta} > 1$ such that $u_n \le \varepsilon^\delta C_{T,\delta}^{n+1}$. We deduce that

$$\mathbb{E}\Big[\sup_{t\in[0,T_n\wedge T]}|v_t^\varepsilon - v_t|^\delta\Big|\mathcal{G}\Big] \le \sum_{k=0}^{n-1} u_k \le u_0 + \dots + u_{n-1} \le \varepsilon^\delta\frac{C_{T,\delta}^{n+1}}{C_{T,\delta} - 1}.$$

Finally,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|v_t^\varepsilon - v_t|^\delta\Big] &\le \sum_{n\ge0}\mathbb{E}\Big[\mathbb{1}_{T_n<T<T_{n+1}}\mathbb{E}\big(\sup_{t\in[0,T_n\wedge T]}|v_t^\varepsilon - v_t|^\delta\big|\mathcal{G}\big)\Big] \\ &\le \frac{\varepsilon^\delta}{C_{T,\delta} - 1}\sum_{n\ge0} C_{T,\delta}^{n+2}\frac{(\lambda T)^n}{n!}e^{-\lambda T},\end{aligned} \tag{2.1.11}$$

with
$$\frac{1}{C_{T,\delta}-1}\sum_{n\geq 0} C_{T,\delta}^{n+2}\frac{(\lambda T)^n}{n!}e^{-\lambda T} < \infty.$$
This gives the result for v^ε. Since
$$x_t^\varepsilon - x_t = \int_0^t (v_s^\varepsilon - v_s)\mathrm{d}s,$$
the proof can be completed by using Jensen's inequality and (2.1.11). □

2.2 Study of the oscillations around the deterministic solution

The study of the oscillations around the deterministic solution is the same for all $\alpha \in (0, 2]$.

Theorem 2.2.1. *For $\beta \geq 1$, we introduce Z, the solution of the differential equation*
$$Z_t = -\int_0^t \mathcal{U}''(v_s)Z_s\mathrm{d}s + \ell_t.$$
Then
$$\frac{1}{\varepsilon}(v^\varepsilon - v - \varepsilon Z) \quad \text{and} \quad \frac{1}{\varepsilon}(x^\varepsilon - x - \varepsilon\int_0^\cdot Z_s\mathrm{d}s)$$
converge UCP toward 0, *as* $\varepsilon \to 0$.

In order to prove the theorem, we need several lemmas.

Lemma 2.2.2. *For all $T \geq 0$, there exists a constant C_{β,T,v_0} depending only on β, T and the initial point of v^ε such that*
$$\sup_{s\in[0,T]}|Z_s| \leq C_{\beta,T,v_0}\sup_{s\in[0,T]}|\ell_s|.$$

Proof. Z is a Lévy Ornstein-Uhlenbeck process starting from 0 so it can be expressed as
$$Z_t = \int_0^t e^{-\int_s^t \mathcal{U}''(v_u)\mathrm{d}u}\mathrm{d}\ell_s.$$
By integration by parts, we obtain:
$$Z_t = \ell_t - \int_0^t \ell_s\mathcal{U}''(v_s)e^{-\int_s^t \mathcal{U}''(v_u)\mathrm{d}u}\mathrm{d}s.$$
Therefore we get, for all $t \leq T$,
$$|Z_t| \leq \sup_{t\in[0,T]}|\ell_t|\big(1 + \sup_{t\in[0,T]}\int_0^t |\mathcal{U}''(v_s)|e^{-\int_s^t \mathcal{U}''(v_u)\mathrm{d}u}\mathrm{d}s\big).$$

Since $\mathcal{U}''$ is always non-negative, we get

$$e^{-\int_s^t \mathcal{U}''(v_u)\mathrm{d}u} \leq 1.$$

Since $|v_s| \leq |v_0|$ and $\mathcal{U}''$ is continuous, we have

$$\sup_{t\in[0,T]} \int_0^t |\mathcal{U}''(v_s)| e^{-\int_s^t \mathcal{U}''(v_u)\mathrm{d}u} \mathrm{d}s \leq T \sup_{y\in[0,v_0]} \mathcal{U}''(y) := C_{\beta,T,v_0} - 1.$$

The lemma is proved. □

Introduce the remainder process $R_t := v_t^\varepsilon - v_t - \varepsilon Z_t$.

Lemma 2.2.3. *For all $T > 0$, there exists a constant C'_{β,T,v_0} depending only on β, T and the initial point of v^ε such that*

$$\sup_{s\in[0,T]} |R_s| \leq C'_{\beta,T,v_0}$$

a.s. on the event

$$\{\sup_{s\in[0,T]} |\varepsilon \ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\}.$$

Proof. By hypothesis, we know that for any $t \in [0, T]$, $|v_T| \leq |v_t| \leq |v_0|$. Moreover, on $\{\sup_{s\in[0,T]} |\varepsilon \ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\}$, due to Lemma 2.2.2, we have

$$\sup_{s\in[0,T]} |\varepsilon Z_s| \leq {}^{|v_T|}\!/_2.$$

Recall that $\beta \geq 1$ so $\mathcal{U}'$ increases at least linearly at infinity. This guarantees the existence of C'_{β,T,v_0} such that for any $y \in [-|v_0|, |v_0|]$, $z \in [-\frac{|v_T|}{2}, \frac{|v_T|}{2}]$, we have :

$$-\mathcal{U}'(y + z + C'_{\beta,T,v_0}) + \mathcal{U}'(y) + \mathcal{U}''(y)z < 0.$$

Hence, we get

$$-\mathcal{U}'(v_t + \varepsilon Z_t + C'_{\beta,T,v_0}) + \mathcal{U}'(v_t) + \mathcal{U}''(v_t)\varepsilon Z_t < 0.$$

on the event $\{\sup_{s\in[0,T]} |\varepsilon \ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\}$. Observe that R satisfies the integral equation

$$R_t = \int_0^t -\mathcal{U}'(v_s + \varepsilon Z_s + R_s) + \mathcal{U}'(v_s) + \mathcal{U}''(v_s)\varepsilon Z_s \mathrm{d}s.$$

So R is continuous and provided that there exists $t_1 < T$ such that $R_{t_1} = C'_{\beta,T,v_0}$, we have

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t_1) = -\mathcal{U}'(v_{t_1} + \varepsilon Z_{t_1} + C'_{\beta,T,v_0}) + \mathcal{U}'(v_{t_1}) + \mathcal{U}''(v_{t_1})\varepsilon Z_{t_1} < 0.$$

We deduce that R decreases and is lower than C'_{β,T,v_0}. A similar reasoning applies for $R_{t_1} = -C'_{\beta,T,v_0}$ and it completes the proof. □

Lemma 2.2.4. *For all $T > 0$, there exists a constant C''_{β,T,v_0} depending only on β, T and the initial point of v^ε such that, on the event $\{\sup_{s\in[0,T]} |\varepsilon\ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\}$, for all $\varepsilon > 0$, all $t < T$, a.s. :*

$$\sup_{s\in[0,T]} |R_s| \leq C''_{\beta,T,v_0}\Big(\sup_{s\in[0,T]} |\varepsilon\ell_s|\Big)^2.$$

Proof. Using Lemmas 2.2.3 and 2.2.2, on the event $\{\sup_{s\in[0,T]} |\varepsilon\ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\}$, for all $t \leq T$, we have

$$|v_T| \leq |v_t| \leq |v_0|, \qquad |\varepsilon Z_t| \leq \frac{|v_T|}{2}, \qquad |R_t| \leq C'_{\beta,T,v_0}.$$

We have

$$\begin{aligned} R_t &= \int_0^t -\mathcal{U}'(v_s + \varepsilon Z_s + R_s) + \mathcal{U}'(v_s) + \mathcal{U}''(v_s)\varepsilon Z_s \mathrm{d}s \\ &= -\int_0^t \big[\mathcal{U}'(v_s+\varepsilon Z_s+R_s)-\mathcal{U}'(v_s+\varepsilon Z_s)\big]\mathrm{d}s-\int_0^t \big[\mathcal{U}'(v_s+\varepsilon Z_s)-\mathcal{U}'(v_s)-\mathcal{U}''(v_s)\varepsilon Z_s\big]\mathrm{d}s. \end{aligned}$$

Choose $K = |v_0| + \frac{|v_T|}{2} + C'_{\beta,T,v_0}$. Since $\mathcal{U}'$ is $\mathcal{C}^1$ on $[-K, K]$ and $\mathcal{U}'$ is $\mathcal{C}^2$ on $[\frac{|v_T|}{2}, |v_0| + \frac{|v_T|}{2}]$, for all $s \leq T$, there exist $\theta_s^1 \in [-K, K]$ and $\theta_s^2 \in [\frac{|v_T|}{2}, |v_0| + \frac{|v_T|}{2}]$ such that

$$\mathcal{U}'(v_s + \varepsilon Z_s + R_s) - \mathcal{U}'(v_s + \varepsilon Z_s) = \mathcal{U}''(\theta_s^1)R_s$$

and

$$\mathcal{U}'(v_s + \varepsilon Z_s) - \mathcal{U}'(v_s) - \mathcal{U}''(v_s)\varepsilon Z_s = \frac{1}{2}\mathcal{U}^{(3)}(\theta_s^2)(\varepsilon Z_s)^2.$$

We deduce

$$|R_t| \leq \int_0^t L|R_s|\mathrm{d}s + \frac{TL(C_{\beta,T,v_0})^2}{2}\Big(\sup_{s\in[0,T]} |\varepsilon\ell_s|\Big)^2,$$

where $L = \max\left(\sup_{[-K,K]} |\mathcal{U}''|, \sup_{[\frac{|v_T|}{2},|v_0|+\frac{|v_T|}{2}]} |\mathcal{U}^{(3)}|\right)$. Using Gronwall's lemma, we complete the proof by setting $C''_{\beta,T,v_0} = \frac{TL(C_{\beta,T,v_0})^2}{2}e^{TL}$. □

Proof of Theorem 2.2.1. We have $\frac{1}{\varepsilon}(v_t^\varepsilon - v_t - \varepsilon Z_t) = \frac{R_t}{\varepsilon}$. If we fix T and δ,

$$\begin{aligned} \mathbb{P}\left(\sup_{s\in[0,T]} \frac{|R_s|}{\varepsilon} \geq \delta\right) &= \mathbb{P}\left(\left\{\sup_{s\in[0,T]} \frac{|R_s|}{\varepsilon} \geq \delta\right\} \cap \left\{\sup_{s\in[0,T]} |\varepsilon\ell_s| \leq \frac{|v_T|}{2C_{\beta,T,v_0}}\right\}\right) \\ &\quad + \mathbb{P}\left(\left\{\sup_{s\in[0,T]} \frac{|R_s|}{\varepsilon} \geq \delta\right\} \cap \left\{\sup_{s\in[0,T]} |\varepsilon\ell_s| > \frac{|v_T|}{2C_{\beta,T,v_0}}\right\}\right) \\ &\leq \mathbb{P}\left(\sup_{s\in[0,T]} \varepsilon\ell_s^2 > \frac{\delta}{C''_{\beta,T,v_0}}\right) + \mathbb{P}\left(\sup_{s\in[0,T]} |\varepsilon\ell_s| > \frac{|v_T|}{2C_{\beta,T,v_0}}\right). \end{aligned}$$

Since $\left\{ \sup_{s\in[0,T]} |\ell_s| \right\}$ is finite a.s., we deduce that

$$\mathbb{P}\left(\sup_{s\in[0,T]} \frac{|R_s|}{\varepsilon} \geq \delta \right) \underset{\varepsilon\to 0}{\to} 0.$$

Finally,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \frac{1}{\varepsilon} \Big| x_t^\varepsilon - x_t - \varepsilon \int_0^t Z_s \mathrm{d}s \Big| > \delta \Big) \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \frac{1}{\varepsilon} |v_t^\varepsilon - v_t - \varepsilon Z_t| > \frac{\delta}{T} \Big)$$

and the proof is complete. ☐

Chapter 3

Asymptotic stability for SDEs driven by symmetric α-stable processes

A slightly different version of this chapter has been published in EJP (see [14]).

We consider the one-dimensional and non-linear Langevin type equation driven by a symmetric α-stable Lévy process. Let us denote by x_t^ε the one-dimensional process describing the position of a particle at time $t \geq 0$, having the speed v_t^ε

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon \mathrm{d}s,\ t \geq 0, \tag{3.0.1}$$

such that v_t^ε is a small symmetric α-stable Lévy process in a potential $\mathcal{U}(x) := \frac{2}{\beta+1}|x|^{\beta+1}$,

$$dv_t^\varepsilon = \varepsilon d\ell_t - \frac{1}{2}\mathcal{U}'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = v_0, \tag{3.0.2}$$

in other words v_t^ε verifies the following integral equation

$$v_t^\varepsilon = v_0 + \varepsilon \ell_t - \int_0^t \mathrm{sgn}(v_s^\varepsilon)|v_s^\varepsilon|^\beta \mathrm{d}s,\ t \geq 0. \tag{3.0.3}$$

Here $\beta > -1$ and $\{\ell_t : t \geq 0\}$ is an α-stable Lévy process, $\alpha \in (0, 2]$. If $\alpha \in (0, 2)$ the Lévy process is a pure jump process with cadlag paths and the jump measure is given by $\nu(\mathtt{d}z) = |z|^{-1-\alpha}\mathbb{1}_{\mathbb{R}\setminus\{0\}}(z)\mathtt{d}z$. The 2-stable Lévy process is the standard Brownian motion $\{b_t : t \geq 0\}$ which is a path continuous process. In all cases, using Theorem 1.1.7, the process has the property of self-similarity $\{\ell_t : t \geq 0\}$ and $\{c^{-1/\alpha}\ell_{ct} : t \geq 0\}$ have the same law, for all $c > 0$.

The case of a harmonic potential ($\beta = 1$, linear equation), when the speed is an Ornstein-Uhlenbeck process, was already considered by Hintze and Pavlyukevich [19]. Here, the authors study the asymptotic behaviour of the integrated Ornstein-Uhlenbeck and prove that this process converges weakly, as $\varepsilon \to 0$, to the underlying α-stable Lévy process. In particular, when the driving process is a Brownian motion ($\alpha = 2$), the asymptotic behaviour is Gaussian. Asymptotics of the first exit time from an interval are deduced. Our goal is to answer the same question in the situation of a super-harmonic potential (non-linear equation): what is the asymptotic behaviour of the the position process x_t^ε, as $\varepsilon \to 0$? On the one hand, the non-linear case introduces new technical difficulties, mainly since the solution is no longer explicit. Indeed, this fact was essential to prove weak convergence in the linear case. On the other hand, different conditions on the two parameters α and β will generate different asymptotics for the position process. The intuition suggests that the big jumps should be compensated by a strong negative drift (for instance if $\beta > 1$) and small jumps should have some regularizing effect. We answer the question by showing that for α and β in some unbounded domain, the position process x_t^ε will behave as a Brownian motion when $\varepsilon \to 0$. In other words, we get Gaussian asymptotic behaviour even if α is smaller than 2, provided that β is not very small, more precisely if $\beta + \frac{\alpha}{2} > 2$. When α and β are in somehow "small" the previous heuristic fails. To get convergence toward a stable process, one needs to change the approach and other technical difficulties appear (this case will be presented in a forthcoming paper, see [18]).

To state the main result of the chapter, we will perform some scaling transformations. Without loss of the generality, we can assume that the initial position is the origin $x_0 = 0$. Moreover, we assume that the initial speed vanishes $v_0 = 0$ since we want to study the asymptotic stability around 0 and the stability, when the initial speed does not vanish, has been made in the Chapter 2. By using the self-similarity, it is clear that the process $\{\mathcal{L}_t^\varepsilon := \varepsilon \ell_{\varepsilon^{-\alpha} t} : t \geq 0\}$ is also an α-stable process. Let us denote, for $t \geq 0$,

$$\mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha} t}^\varepsilon \quad \text{and} \quad \mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha} t}^\varepsilon \tag{3.0.4}$$

satisfying, respectively,

$$\mathcal{X}_t^\varepsilon = \frac{1}{\varepsilon^\alpha} \int_0^t \mathcal{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \mathcal{V}_t^\varepsilon = \mathcal{L}_t^\varepsilon - \frac{1}{\varepsilon^\alpha} \int_0^t \operatorname{sgn}(\mathcal{V}_s^\varepsilon) |\mathcal{V}_s^\varepsilon|^\beta \mathrm{d}s. \tag{3.0.5}$$

To simplify the notations, all along the chapter we will set

$$\theta = \theta_{\alpha,\beta} := \frac{\alpha}{\alpha + \beta - 1} > 0, \quad \text{provided that } \alpha + \beta - 1 > 0. \tag{3.0.6}$$

Moreover we introduce

$$L_t^\varepsilon := \frac{\mathcal{L}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta} = \frac{\ell_{t\,\varepsilon^{-(\beta-1)\theta}}}{\varepsilon^{(\beta-1)\theta/\alpha}} \quad \text{and} \quad V_t^\varepsilon := \frac{\mathcal{V}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta}, \tag{3.0.7}$$

By self-similarity again, L^ε is distributed as an α-stable Lévy process and we have

$$\mathcal{X}_t^\varepsilon = \varepsilon^{(2-\beta)\theta} \int_0^{t\varepsilon^{-\alpha\theta}} V_s^\varepsilon \mathrm{d}s \quad \text{and} \quad V_t^\varepsilon = L_t^\varepsilon - \int_0^t \operatorname{sgn}(V_s^\varepsilon) |V_s^\varepsilon|^\beta \mathrm{d}s. \tag{3.0.8}$$

Let us note that if $\alpha = 2$, all previous computations hold true with ℓ, $\mathcal{L}$ or L replaced respectively by b, $\mathcal{B}$ or B a standard Brownian motion.

Our main result is the following:

Theorem 3.0.5. *1. (Brownian driving noise) Assume that $\alpha = 2$, $\beta > -1$ and recall that $\theta = \frac{2}{\beta+1}$. There exists a positive constant $\kappa_{2,\beta}$ such that the process*

$$\{\varepsilon^{(\beta-1)\theta} x_{\varepsilon^{-2} t}^\varepsilon : t \geq 0\} = \{\varepsilon^{(\beta-1)\theta} \mathcal{X}_t^\varepsilon : t \geq 0\} \tag{3.0.9}$$

converges in distribution, in the space of continuous functions $\mathrm{C}([0,\infty))$ *endowed with the uniform topology on compact sets, to a Brownian motion process with variance* $\kappa_{2,\beta}$*, as* $\varepsilon \to 0$*. The constant* $\kappa_{2,\beta}$ *has the integral representation given in* (3.1.9) *below.*

2. *(symmetric stable driving noise) Assume that* $\alpha \in (0,2)$*,* $\beta + \frac{\alpha}{2} > 2$ *and recall that* $\theta = \frac{\alpha}{\alpha+\beta-1}$*. There exists a positive constant* $\kappa_{\alpha,\beta}$ *such that the process*

$$\{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta} x^{\varepsilon}_{\varepsilon^{-\alpha}t} : t \geq 0\} = \{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta} \mathcal{X}^{\varepsilon}_t : t \geq 0\} \tag{3.0.10}$$

converges in distribution, in the space of continuous functions $\mathrm{C}([0,\infty))$ *endowed with the uniform topology on compact sets, to a Brownian motion process with variance* $\kappa_{\alpha,\beta}$*, as* $\varepsilon \to 0$*. The constant* $\kappa_{\alpha,\beta}$ *has the integral representation given in* (3.2.34) *below.*

Remark 3.0.6. 1. *Hypotheses* $\alpha \in (0,2)$ *and* $\beta + \frac{\alpha}{2} > 2$ *imply that* $\beta > 1$*, in other words the drift is over-damped. In particular we have* $\theta \in (0,1)$*.*

2. *If the driving noise is the Brownian motion* $\alpha = 2$*, the normalizing factor behaves differently following with the position of* β *with respect to 1. If* $\beta = 1$*, the position process* $\mathcal{X}^{\varepsilon}$ *converges in distribution to a standard Brownian motion, see also* Remark 3.1.4 *below.*

3. *The case when* $\beta + \frac{\alpha}{2} = 2$ *should be considered as critical for some phase transition from Gaussian to stable case. It should be reasonable that there is some continuity but the proof seems more delicate since natural integrability conditions are not fulfilled (see the method of proof described below).*

4. *As an application one can find asymptotics of the first exit time from an interval: Corollary* 2.1*, p. 269, in* [19] *applies.*

Let us explain the method of the proof and the organization of the chapter. It is a simple observation that

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} \mathcal{X}^{\varepsilon}_t = \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} V^{\varepsilon}_s \mathrm{d}s\,,$$

hence, since L^{ε} is a symmetric α-stable process, V^{ε} is zero mean and Theorem 3.0.5 is a second order type ergodic theorem. Let us only note that in the asymmetric case, a drift term appears in the Lévy-Itô decomposition of the driving noise. Consequently, the expression of the infinitesimal generator of V^{ε} will be different and finally a first order term should appear in the limit. In this case, Theorem 1.3.3 will not apply and we will see in Chapter 4 how to obtain a result. By using the stochastic calculus, we will show that the latter quantity is the sum of a square integrable martingale, provided that $\beta + \frac{\alpha}{2} > 2$ for the case $\alpha \in (0,2)$, and a term which tends in probability toward 0, as $\varepsilon \to 0$. The result is then obtained by using the functional central limit theorem for martingales and the continuous-mapping theorem. In the critical case

$\beta + \frac{\alpha}{2} = 2$ and $\alpha \in (0,2)$, the L^2-integrability fails. We point out that the critical case for the Brownian noise is the case studied in [19].

In the next section, we consider the case when the driving noise is the Brownian motion: in this case computations are performed by using Itô's calculus and are more explicit. For instance, the constant $\kappa_{2,\beta}$ can be written in terms of the scale function and the speed measure. In Section 2, we follow the same structure of the proof for a pure jump driving noise. Computations are more technical and new ideas are employed: for instance, we need to find and use a Lyapunov function which allows to perform the same reasoning by using the Itô-Lévy calculus.

3.1 Brownian case

Recall that in this case, $\{b_t : t \geq 0\}$ is a standard one-dimensional Brownian motion, $\beta > -1$ and we set

$$\check{B}_t^\varepsilon := \frac{B^\varepsilon_{t\,\varepsilon^{4/(\beta+1)}}}{\varepsilon^{2/(\beta+1)}} = \frac{b_{t\,\varepsilon^{2(1-\beta)/\beta+1}}}{\varepsilon^{(\beta-1)/(\beta+1)}}, \quad \text{and} \quad \check{V}_t^\varepsilon := \frac{\mathcal{V}^\varepsilon_{t\,\varepsilon^{4/(\beta+1)}}}{\varepsilon^{2/(\beta+1)}}. \tag{3.1.1}$$

Recall also that

$$\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{2(2-\beta)}{(\beta+1)}} \int_0^{t\varepsilon^{-4/(\beta+1)}} \check{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \check{V}_t^\varepsilon = \check{B}_t^\varepsilon - \int_0^t \operatorname{sgn}(\check{V}_s^\varepsilon)|\check{V}_s^\varepsilon|^\beta \mathrm{d}s\,. \tag{3.1.2}$$

$\check{B}^\varepsilon$ is distributed as a standard Brownian motion so, to simplify the notation, we will suppress the index ε, as well as for $\check{V}^\varepsilon$.

3.1.1 The speed process V^ε

If $\beta \geq 1$, the drift coefficient in $(3.1.2_2)$ is a locally Lipschitz function hence by Theorem 1.2.7, we get a pathwise unique strong solution $\check{V}$ to equation $(3.1.2_2)$, whereas in the case $-1 < \beta < 1$, Girsanov's theorem gives the existence of a weak solution to equation $(3.1.2_2)$. For both situations, the solution is defined until an explosion time τ_e. Introduce the scale function and the speed measure associated to the diffusion

$$s_\beta(x) := \int_0^x \mathrm{e}^{-c_\beta(y)}\mathrm{d}y \quad \text{and} \quad m_\beta(\mathrm{d}x) := 2\mathrm{e}^{c_\beta(x)}\mathrm{d}x, \quad \text{where} \quad c_\beta(x) := -\frac{2}{\beta+1}|x|^{\beta+1}\,. \tag{3.1.3}$$

Since $\int_0^\infty m_\beta([0,x])\mathrm{e}^{-c_\beta(x)}\mathrm{d}x = \infty$, by Theorem 1.2.9, the solution is global and finally, there exists a pathwise unique strong solution $\check{V}$ to the equation $(3.1.2_2)$.

Convergence in probability of the Speed process

We study first the speed process.

Proposition 3.1.1. *As $\varepsilon \to 0$, $\{V_t^\varepsilon : t \geq 0\}$ is convergent toward 0 in probability uniformly on each compact interval.*

By $(3.1.1_2)$, the relation between V^ε and $\check{V}$ is $V_t^\varepsilon = \varepsilon^{2/(\beta+1)} \check{V}_{t\,\varepsilon^{-4/(\beta+1)}}$. To prove Proposition 3.1.1, we need a preliminary result:

Lemma 3.1.2.

1. *Fix $p \geq 2$. There exists a positive constant $C_{p,\beta}$ such that, for any $t \geq 0$,*

$$\mathbb{E}\Big(|\check{V}_t|^p\Big) \leq C_{p,\beta}\, t. \tag{3.1.4}$$

2. *Fix $p \geq 4$ and $T > 0$. There exists a positive constant $C'_{p,\beta}$ such that*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|\Big)^p \leq C'_{p,\beta}\, T^2\, \varepsilon^{-8/(\beta+1)}. \tag{3.1.5}$$

Proof of Proposition 3.1.1. Taking $p > 4$ in Lemma 3.1.2, we deduce that for any $T > 0$, as $\varepsilon \to 0$, $\sup_{0\leq t\leq T} |V_t^\varepsilon|$ converges to 0 in $\mathrm{L}^p(\Omega)$, and the conclusion follows. $\square$

Proof of Lemma 3.1.2. By using Itô's formula and the equation $(3.1.2_2)$, we can write

$$|\check{V}_t|^p = p\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s + p\int_0^t \Big((1/2)(p-1)|\check{V}_s|^{p-2} - |\check{V}_s|^{p-1+\beta}\Big)\mathrm{d}s$$

Since $\beta > -1$, there exists a constant $C_{p,\beta} > 0$ such that

$$p\Big((1/2)(p-1)|x|^{p-2} - |x|^{p-1+\beta}\Big) \leq C_{p,\beta},\ \forall x \in \mathbb{R}.$$

We deduce that

$$|\check{V}_t|^p \leq C_{p,\beta} t + p\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s \tag{3.1.6}$$

We show that $\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s$ is a martingale. Fix $T > 0$, for all $t \leq T$, since $(a+b)^2 \leq 2(a^2+b^2)$ and $|x|^{2p-2} \leq 1 + |x|^{2p}$, by using the Burkholder-Davis-Gundy inequality, we can see that there exists a positive constant C'_1 such that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq u\leq t} |\check{V}_u|^p\Big)^2 &\leq 2C_{p,\beta}^2\, T^2 + 2p^2\, \mathbb{E}\Big(\sup_{0\leq u\leq t}\int_0^u \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s\Big)^2 \leq 2C_{p,\beta}^2\, T^2 \\ &+ 2p^2\, C'_1 \int_0^t \mathbb{E}\big(|\check{V}_s|^{2p-2}\big)\mathrm{d}s \leq 2p^2\, C'_1\, T + 2C_{p,\beta}^2\, T^2 + 2p^2\, C'_1\int_0^t \mathbb{E}\big(|\check{V}_s|^{2p}\big)\mathrm{d}s \\ &\leq 2p^2\, C'_1\, T + 2C_{p,\beta}^2\, T^2 + 2p^2\, C'_1 \int_0^t \mathbb{E}\big(\sup_{0\leq u\leq s}|\check{V}_u|^p\big)^2\mathrm{d}s.\end{aligned}$$

By Gronwall's lemma, we get, for all $t \leq T$,

$$\mathbb{E}\Big(\sup_{0\leq u\leq t}|\check{V}_u|^p\Big)^2 \leq (2p^2\, C'_1\, T + 2C_{p,\beta}^2\, T^2)\mathrm{e}^{2p^2\, C'_2\, T}.$$

Hence $\int_0^t \operatorname{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s$ is a martingale and we get (3.1.4) by taking expectation in (3.1.6).

It is now possible to improve this inequality. Using (3.1.4), we can see that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}|\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|\Big)^p &= \mathbb{E}\Big(\sup_{0\le t\le T}|\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|^{p/2}\Big)^2 \\
&\le \frac{p^2}{2}\,\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^{t\,\varepsilon^{-4/(\beta+1)}}|\check{V}_s|^{p/2-1}\mathrm{d}\check{B}_s\Big)^2 + 2\,C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)} \\
&\le \frac{p^2}{2}C_1'\int_0^{T\,\varepsilon^{-4/(\beta+1)}}\mathbb{E}\big(|\check{V}_s|^{p-2}\big)\mathrm{d}s + 2C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)} \\
&\le \frac{p^2}{4}C_1'\,C_{p-2,\beta}\,T^2\,\varepsilon^{-8/(\beta+1)} + 2C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)}.
\end{aligned}$$

Therefore (3.1.5) follows taking $C_{p,\beta}' := \frac{p^2}{4}C_1'\,C_{p-2,\beta} + 2C_{p/2,\beta}^2$. □

3.1.2 The position process $\mathcal{X}^\varepsilon$

Let us now study the position process. Recall that we introduced the scale function and the speed measure in (3.1.3). Since $s_\beta(\infty) = \infty$ and $m_\beta(\mathbb{R}) < \infty$, by Proposition 1.3.1, the diffusion $\check{V}$ is regular and is a recurrent and ergodic process with the invariant measure $\frac{1}{m_\beta(\mathbb{R})}m_\beta$. Therefore, for all $f \in \mathrm{L}^1(m_\beta)$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\check{V}_s)\mathrm{d}s = \frac{1}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} f(x)m_\beta(\mathrm{d}x),\ \text{ almost surely.} \tag{3.1.7}$$

We recall that the infinitesimal generator of $\check{V}$ is given by $\mathcal{L}_{2,\beta} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \operatorname{sgn}(x)|x|^\beta\frac{\mathrm{d}}{\mathrm{d}x}$. Introduce

$$g_\beta(x) := \int_0^x\left(\int_y^{+\infty} -2z\mathrm{e}^{c_\beta(z)}\mathrm{d}z\right)\mathrm{e}^{-c_\beta(y)}\mathrm{d}y,\ x\in\mathbb{R}, \tag{3.1.8}$$

and note that $(\mathcal{L}_{2,\beta}\,g_\beta)(x) = x$, for all $x\in\mathbb{R}$. Set

$$\kappa_{2,\beta} := \frac{1}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} g_\beta'(x)^2 m_\beta(\mathrm{d}x) = -\frac{2}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} x g_\beta'(x)m_\beta(\mathrm{d}x). \tag{3.1.9}$$

We can give now the proof of the main result.

Proof of Theorem 3.0.5 in the case $\alpha = 2$. By applying Itô's formula, we can see that

$$g_\beta(\check{V}_t) = \int_0^t g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \int_0^t(\mathcal{L}_{2,\beta}\,g_\beta)(\check{V}_s)\mathrm{d}s = \int_0^t g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \int_0^t \check{V}_s\mathrm{d}s,$$

and therefore

$$\varepsilon^{2(\beta-1)/(\beta+1)}\mathcal{X}_t^\varepsilon = -\varepsilon^{2/(\beta+1)}\int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \varepsilon^{2/(\beta+1)}g_\beta(\check{V}_{t\,\varepsilon^{-4/(\beta+1)}})\,.$$

The continuous local martingale

$$M_t^\varepsilon := -\varepsilon^{2/(\beta+1)} \int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s$$

has the quadratic variation

$$\langle M^\varepsilon \rangle_t = \varepsilon^{4/(\beta+1)} \int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check{V}_s)^2 \mathrm{d}s\,.$$

As a consequence of (3.1.7), for all t, $\langle M^\varepsilon \rangle_t \to \kappa_{2,\beta} t$ a.s. as $\varepsilon \to 0$, where $\kappa_{2,\beta}$ is given by (3.1.9), and it is the constant in the statement of Theorem 3.0.5. Indeed, using Whitt's theorem (see Theorem 2.1(ii), p. 270 in [45]), we deduce that M^ε converges in distribution (as a process) toward $\kappa_{2,\beta}^{1/2}\check{B}$.

We will prove that the second term in the right hand side converges in probability uniformly on compact sets to 0. At this level, we need a technical result:

Lemma 3.1.3. *There exist two positive constants μ_β, ν_β such that for all $x \in \mathbb{R}$,*

$$|g_\beta(x)| \le \mu_\beta |x|^{(2-\beta)\vee 1} + \nu_\beta. \tag{3.1.10}$$

We postpone the proof of the lemma and finish the proof of Theorem 3.0.5 in the case $\alpha = 2$. By using the classical inequality $(a+b)^{2m} \le 2^{2m-1}(a^{2m} + b^{2m})$, ($m \ge 1$ integer), we obtain

$$|\varepsilon^{2/(\beta+1)} g_\beta(\check{V}_{t\,\varepsilon^{-4/(\beta+1)}})|^{2m} \le 2^{2m-1}\mu_\beta^{2m}\varepsilon^{(4m)/(\beta+1)}|\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|^{2m((2-\beta)\vee 1)} + 2^{2m-1}\nu_\beta^{2m}\varepsilon^{(4m)/(\beta+1)}\,.$$

By choosing the integer $m \ge 2$ such that $p := 2m((2-\beta)\vee 1) > 4$, we can use Lemma 3.1.2 and we get for all $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{0\le t\le T} \varepsilon^{4m/(\beta+1)} g_\beta^{2m}(\check{V}_{t\,\varepsilon^{-4/(\beta+1)}})\Big] = 0.$$

We finish the proof of the theorem by employing the joint convergence theorem and the simple continuous-mapping theorem (Theorem 11.4.5 p. 379 and Theorem 3.4.1, p. 85 in [46]) on the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology. $\square$

Proof of Lemma 3.1.3. We note that g_β is an odd function. Let us introduce the function $\varphi_\beta(x) = -\int_x^{+\infty} 2y\mathrm{e}^{c_\beta(y)}\mathrm{d}y$. By the continuity of g_β on $[0,1]$, it is sufficient to prove (3.1.10) for $x > 1$. Assume that $\beta \in [1,\infty)$. Then, since $x > 1$,

$$\varphi_\beta(x) = \int_x^{+\infty} z^{1-\beta}\left(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\right)\mathrm{d}z \ge \int_x^{+\infty} -2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\mathrm{d}z = -\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}},$$

hence

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y \ge 1 - x.$$

(3.1.10) is true in this case. If $\beta \in [0,1)$, by integration by parts,

$$\varphi_\beta(x) = -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} + \frac{1-\beta}{2}\int_x^{+\infty} z^{-2\beta}\big(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\big)\mathrm{d}z$$
$$\geq -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} - \frac{1-\beta}{2}x^{-2\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}.$$

Hence,

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y \geq \int_1^x \left(-y^{1-\beta} - \frac{1-\beta}{2}y^{-2\beta}\right)\mathrm{d}y,$$

and (3.1.10) follows. More generally, assume that $\beta \in [-\frac{n}{n+2}, \frac{1-n}{n+1})$, for an integer $n \geq 0$. Set $d_0 = 1$ and $d_k := 2^{-k}\prod_{j=0}^{k-1}((1-\beta) - j(1+\beta))$, for $k \geq 1$ integer. By the choice of n, we can see that $d_n > 0$. If we iterate n times the integration by parts, we get:

$$\varphi_\beta(x) = -\sum_{k=0}^n d_k x^{(1-\beta)-k(1+\beta)}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} + d_n\int_x^{+\infty} z^{(1-\beta)-(n+1)(\beta+1)}(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}})\mathrm{d}z.$$

Since $(1-\beta)-(n+1)(\beta+1) \leq 0$, we can write

$$\varphi_\beta(x) \geq -\left(\sum_{k=0}^n d_k x^{(1-\beta)-k(1+\beta)} + d_n x^{(1-\beta)-(n+1)(\beta+1)}\right)\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}.$$

By integrating, we have

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y \geq \int_1^x \left(\sum_{k=0}^n d_k y^{(1-\beta)-k(1+\beta)} + d_n y^{(1-\beta)-(n+1)(\beta+1)}\right)\mathrm{d}y,$$

and we easily deduce (3.1.10). The proof of (3.1.10) is complete for all $\beta \in (-1,\infty)$. □

Remark 3.1.4. *Let us note that if $\beta = 1$ (Ornstein-Uhlenbeck case), $g_\beta(x) = -x$, $\kappa_{2,\beta} = 1$ and the result of Theorem* 3.0.5 *coincides with the result of Proposition* 2.1 *in* [19].

3.2 Pure jump case

We recall that L^ε is distributed as a α-stable Lévy process (see (3.0.7$_1$)) so, to simplify the notation, we will suppress the index ε, as well as for V^ε (see (3.0.8$_2$)). The drift satisfies the condition of the Proposition 1.2.10 so there exists a global path-wise unique strong solution V for equation (3.0.8$_2$)

The ergodic feature of the process V is a consequence of Theorem 1.3.2. Indeed, provided that $\beta > 1$, the drift coefficient $b(x) = -\mathrm{sgn}(x)|x|^\beta$ and the jump measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\mathbb{1}_{\mathbb{R}\setminus\{0\}}\mathrm{d}z$ clearly satisfy the conditions in the cited result. Hence V

is an exponential ergodic and Harris recurrent process having an unique invariant distribution which is symmetric, denoted by $m_{\alpha,\beta}$. The measure $m_{\alpha,\beta}$ satisfies

$$m_{\alpha,\beta}([x,+\infty)) \underset{x\to\infty}{\sim} \int_x^{+\infty} \frac{\nu([u,+\infty))}{-b(x)} \mathrm{d}u = \frac{C}{x^{\alpha+\beta-1}}\,, \tag{3.2.11}$$

as it follows from Theorem 1.3.3.

Clearly, under the hypothesis of Theorem 3.0.5, $\beta + \frac{\alpha}{2} - 2 > 0$, the identity function id belongs to $\mathrm{L}^1(m_{\alpha,\beta})$. By the classical ergodic theorem, for all $f \in \mathrm{L}^1(m_{\alpha,\beta})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(V_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) m_{\alpha,\beta}(\mathrm{d}x), \quad \text{a.s.} \tag{3.2.12}$$

Recall that we are interested on the behaviour as $\varepsilon \to 0$ of

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} x^{\varepsilon}_{\varepsilon^{-\alpha}t} = \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} V_s \mathrm{d}s, \tag{3.2.13}$$

where θ is given by (3.0.6). In other words, we are studying a large time behaviour of a functional of V, hence it is quite natural to perform the study in steady state. This fact is contained in the following lemma:

Lemma 3.2.1. *Assume that $\beta+\frac{\alpha}{2}-2>0$ and that the process $\left(\varepsilon^{\alpha\theta/2}\int_0^{t\varepsilon^{-\alpha\theta}} V_s\mathrm{d}s\right)_{t\geq 0}$ converges, as $\varepsilon \to 0$, in distribution to a Brownian motion, provided that V is starting with $m_{\alpha,\beta}$ as an initial distribution. Then the same process converges in distribution to a Brownian motion when $V_0 = 0$.*

Proof. In this proof we will denote the process in (3.2.13) by $Z_{\varepsilon,0}(t)$, and for $\Delta \geq 0$,

$$Z_{\varepsilon,\Delta}(t) := \varepsilon^{\frac{\alpha\theta}{2}} \int_{\Delta}^{t\varepsilon^{-\alpha\theta}+\Delta} V_s\mathrm{d}s.$$

Firstly, let us prove that $Z_{\varepsilon,\Delta}(\cdot)$ converges in distribution, as $\Delta \to \infty$ and $\varepsilon \to 0$, to a Brownian motion, when $V_0 = 0$. Denoting by μ_Δ the distribution of V_Δ, for each bounded continuous real function ψ on $\mathrm{C}([0,+\infty))$, by the Markov property, we have

$$\mathbb{E}_0\big[\psi(Z_{\varepsilon,\Delta}(\cdot))\big] = \mathbb{E}_{\mu_\Delta}\big[\psi(Z_{\varepsilon,0}(\cdot))\big].$$

We can write, for all $\varepsilon > 0$,

$$\begin{aligned}
&\Big|\mathbb{E}_{\mu_\Delta}\big[\psi(Z_{\varepsilon,0}(\cdot))\big] - \mathbb{E}_{m_{\alpha,\beta}}\big[\psi(Z_{\varepsilon,0}(\cdot))\big]\Big| \\
&\qquad = \Big|\int_{\mathbb{R}} \mathbb{E}_y\big[\psi(Z_{\varepsilon,0}(\cdot))\big]\big(\mu_\Delta(\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big)\Big| \\
&\quad \leq ||\psi||_\infty \int_{\mathbb{R}} \big|p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big| \leq ||\psi||_\infty ||p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)||_{\mathrm{TV}}\,,
\end{aligned}$$

where $p(t,x,\mathrm{d}y) = \mathbb{P}_x(V_t \in \mathrm{d}y)$ is the transition kernel of V (and therefore we have $p(\Delta,0,\mathrm{d}y) = \mu_\Delta(\mathrm{d}y)$), and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Since V is exponentially ergodic, we get that

$$\lim_{\Delta\to\infty} \big|\mathbb{E}_{\mu_\Delta}\big[\psi(Z_{\varepsilon,0}(\cdot))\big] - \mathbb{E}_{m_{\alpha,\beta}}\big[\psi(Z_{\varepsilon,0}(\cdot))\big]\big| = 0, \qquad \text{uniformly in } \varepsilon.$$

Secondly, by choosing $\Delta = \Delta(\varepsilon) = \varepsilon^{-\alpha\theta/4}$ we obtain

$$\sup_{t\geq 0}\left\{\left|Z_{\varepsilon,\Delta(\varepsilon)}(t) - \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{t\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s\mathrm{d}s\right|\right\} \leq \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s = \varepsilon^{\frac{\alpha\theta}{4}}\frac{1}{\Delta(\varepsilon)}\int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s.$$

The right hand side term of the latter inequality tends to 0 almost surely, by using the ergodicity (3.2.12). Therefore $\varepsilon^{\alpha\theta/2}\int_0^{\bullet\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s\mathrm{d}s$ converges in distribution, as $\varepsilon \to 0$, to a Brownian motion when $V_0 = 0$. Clearly, $\lim_{\varepsilon\to 0}(t - \Delta(\varepsilon)\varepsilon^{\alpha\theta}) = t$, and applying a consequence of the continuous mapping theorem for the composition function stated in Lemma p. 151 in [6], we can conclude. $\square$

In the sequel, we will always assume that the initial distribution of V is $m_{\alpha,\beta}$. Let us recall that the infinitesimal generator of V is given by

$$(\mathcal{A}_{\alpha,\beta}\, g)(x) = -\mathrm{sgn}(x)|x|^\beta g'(x) + \int_{\mathbb{R}}\Big[g(x+y) - g(x) - yg'(x)\mathbb{1}_{|y|\leq 1}\Big]\nu(\mathrm{d}y)\,, \quad (3.2.14)$$

with the domain $D_{\mathcal{A}_{\alpha,\beta}}$. Also, denote by $(\mathcal{T}_t)_{t\geq 0}$ the semi-group associated to the operator $\mathcal{A}_{\alpha,\beta}$ or to the process V. We collect in the following lemma some useful properties of the process V.

Lemma 3.2.2.

1. *The domain $D_{\mathcal{A}_{\alpha,\beta}}$ contains the space of bounded twice differentiable functions $\mathrm{C}_b^2(\mathbb{R})$.*
2. *For all $p \geq 1$, $\mathcal{T}_t$ is a contraction semi-group on $\mathrm{L}^p(m_{\alpha,\beta})$ and for each $f \in \mathrm{L}^p(m_{\alpha,\beta})$,*

$$\lim_{t\to 0}\|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} = 0. \quad (3.2.15)$$

Proof. To prove the first point, we fix $f \in \mathrm{C}_b^2(\mathbb{R})$ and we show that $\mathcal{A}_{\alpha,\beta} f$ is well defined. Let us note that, $-\mathrm{sgn}(x)|x|^\beta f'(x)$ is well defined for all $x \in \mathbb{R}$. Since $f \in \mathrm{C}_b^2(\mathbb{R})$, for any $y \in [-1, 1]$,

$$\Big|f(x+y) - f(x) - yf'(x)\Big| \leq y^2 \sup_{z\in[x-1,x+1]} |f''(z)| < \infty,$$

and we find

$$\int_{|y|\leq 1}\Big[f(x+y) - f(x) - yf'(x)\Big]\nu(\mathrm{d}y) \leq \Big[\sup_{z\in[x-1,x+1]}|f''(z)|\Big]\int_{|y|\leq 1} y^2\nu(\mathrm{d}y) < \infty.$$

Since f is bounded, we can see that

$$\int_{|y|>1}\Big[f(x+y) - f(x)\Big]\nu(\mathrm{d}y) \leq 2\|f\|_\infty \int_{|y|>1}\nu(\mathrm{d}y) < \infty\,,$$

hence $f \in \mathcal{D}_{\mathcal{A}_{\alpha,\beta}}$.

We proceed with the proof of the second point. Fix $f \in \mathrm{L}^p(m_{\alpha,\beta})$ and we show first that

$$\|\mathcal{T}_t f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \|f\|_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Since

$$\|\mathcal{T}_t f\|^p_{\mathrm{L}^p(m_{\alpha,\beta})} = \int_{\mathbb{R}} |\mathcal{T}_t f(x)|^p m_{\alpha,\beta}(\mathrm{d}x) = \int_{\mathbb{R}} |\mathbb{E}_x(f(V_t))|^p m_{\alpha,\beta}(\mathrm{d}x),$$

by Jensen's inequality ($p \geq 1$), we get

$$\|\mathcal{T}_t f\|^p_p \leq \int_{\mathbb{R}} \mathbb{E}_x(|f(V_t)|^p) m_{\alpha,\beta}(\mathrm{d}x) = \mathbb{E}_{m_{\alpha,\beta}}(|f(V_t)|^p) = ||f||^p_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Finally, we prove (3.2.15). Since $\mathrm{C}^2_b(\mathbb{R})$ is dense in $\mathrm{L}^p(m_{\alpha,\beta})$, there exists $f_\eta \in \mathrm{C}^2_b(\mathbb{R})$ such that $||f - f_\eta||_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \eta/3$. Since $\mathcal{T}_t$ is a contraction semi-group and $m_{\alpha,\beta}$ is a probability measure, we get

$$\|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq 2\|f - f_\eta\|_{\mathrm{L}^p(m_{\alpha,\beta})} + \|\mathcal{T}_t f_\eta - f_\eta\|_\infty \leq (2\eta)/3 + \|\mathcal{T}_t f_\eta - f_\eta\|_\infty.$$

Clearly $\mathcal{T}_t$ is a Feller semi-group (see for instance [1], p. 151). Hence $\|\mathcal{T}_t f_\eta - f_\eta\|_\infty \leq \eta/3$, for t small enough, and we deduce (3.2.15). The proof is complete. □

3.2.1 Properties of the speed process

One describes the behaviour of the speed process by using a Lyapunov function. The statement of this important result is given below.

Proposition 3.2.3. *Suppose that $\beta + \frac{\alpha}{2} > 2$ and let p and γ such that*

$$p > 1, \;\; p\gamma > 2, \;\; 2 - \beta < \gamma < \frac{\alpha}{2}. \tag{3.2.16}$$

Introduce the Lyapunov function

$$h_{p,\gamma}(x) := (1 + |x|^{p\gamma})^{1/p}. \tag{3.2.17}$$

Then, as $\varepsilon \to 0$, $\{\varepsilon^{\alpha\theta/2} h_{p,\gamma}(\varepsilon^{-\theta}\mathcal{V}^\varepsilon_t) : t \geq 0\}$ converges to 0 in probability, uniformly on each compact time interval. More precisely, there exists $q > 2$ such that, for any fixed $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\varepsilon^{-\theta}\mathcal{V}^\varepsilon_t\Big)\Big)^q\Big] = \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = 0. \tag{3.2.18}$$

In order to prove this result, we need the following lemma. The first part of the lemma collects some regularity properties and the asymptotic behaviour of the Lyapunov function, while the second part contains the Foster-Lyapunov conditions which allows to solve Poisson's equations.

Lemma 3.2.4.

1. *If $p\gamma > 2$, $h_{p,\gamma}$ is a twice differentiable function and there exists a positive constant k such that for all $(x, y) \in \mathbb{R}^2$,*

- *if* $|x| < 1$ *then*

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \le k(|y|\mathbb{1}_{\{|y|\le 1\}} + |y|^\gamma \mathbb{1}_{\{|y|>1\}});$$

- *if* $|x| \ge 1$ *then*

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \le k(|y||x|^{\gamma-1}\mathbb{1}_{\{|y|\le i(x)\}} + |y|^\gamma \mathbb{1}_{\{i(x)<|y|\}}),$$

where $i(x) := (2|x|^{p\gamma}+1)^{1/p\gamma} - |x|$.

2. *Assume that* $p\gamma > 2$ *and* $2-\beta < \gamma < \alpha$. *There exist a continuous function* $f_{p,\alpha,\beta,\gamma}$, *a compact set* K *and a constant* d, *depending only on* p, α, β, γ, *such that*

$$\forall x \in \mathbb{R},\ f_{p,\alpha,\beta,\gamma}(x) \ge 1+|x|, \quad f_{p,\alpha,\beta,\gamma}(x) \underset{|x|\to\infty}{\sim} \gamma|x|^{\gamma+\beta-1}, \tag{3.2.19}$$

and

$$(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x) \le -f_{p,\alpha,\beta,\gamma}(x) + d\,\mathbb{1}_K. \tag{3.2.20}$$

Proof of Lemma 3.2.4. Recall that $h_{p,\gamma}(x) = (1+|x|^{p\gamma})^{1/p}$ and assume firstly that $|x|<1$. Since $h_{p,\gamma}$ is continuously differentiable and equivalent to $|x|^\gamma$, as $|x|\to\infty$, there exists $k>0$ such that

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \le |y| \sup_{z\in[-2,2]} |h'_{p,\gamma}(z)|\mathbb{1}_{\{|y|\le 1\}} + k|y|^\gamma \mathbb{1}_{\{|y|>1\}}.$$

The desired inequality is then clear. Secondly, assume that $|x|\ge 1$. It is a simple computation to see that for all $z\ge 0$ and $r>0$, there exists $c_r>0$, such that

$$(1+z)^r - 1 \le c_r\big(z\mathbb{1}_{\{z\le 1\}} + z^r\mathbb{1}_{\{z>1\}}\big).$$

We deduce that, for all $(u,v)\in[0,\infty)\times[0,\infty)$, there exist $k_r>0$ such that

$$(u+v)^r - u^r = u^r\left[\left(1+\frac{v}{u}\right)^r - 1\right] \le k_r\big(vu^{r-1}\mathbb{1}_{\{v\le u\}} + v^r\mathbb{1}_{\{u<v\}}\big). \tag{3.2.21}$$

Since $x\neq 0$,

$$\begin{aligned}|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| &= |x|^\gamma \left|\left(\frac{1}{|x|^{p\gamma}} + \left|1+\frac{y}{x}\right|^{p\gamma}\right)^{1/p} - \left(\frac{1}{|x|^{p\gamma}}+1\right)^{1/p}\right| \\ &\le |x|^\gamma\left[\left(\frac{1}{|x|^{p\gamma}} + \left(1+\left|\frac{y}{x}\right|\right)^{p\gamma}\right)^{1/p} - \left(\frac{1}{|x|^{p\gamma}}+1\right)^{1/p}\right].\end{aligned}$$

Applying (3.2.21) with $u = \frac{1}{|x|^{p\gamma}}+1$, $v = \left(1+\left|\frac{y}{x}\right|\right)^{p\gamma} - 1$ and $r=\frac{1}{p}$, we obtain

$$\begin{aligned}&|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \\ &\quad\le k_{1/p}|x|^\gamma\left[\left(\left(1+\left|\frac{y}{x}\right|\right)^{p\gamma}-1\right)^{1/p}\mathbb{1}_{\{i(x)\le|y|\}}\right. \\ &\qquad\left. + \left(\left(1+\left|\frac{y}{x}\right|\right)^{p\gamma}-1\right)\left(\frac{1}{|x|^{p\gamma}}+1\right)^{\frac{1-p}{p}}\mathbb{1}_{\{|y|<i(x)\}}\right].\end{aligned}$$

Since $i(x) > |x|$, we can use again (3.2.21) to estimate the first term in the bracket on the right hand of the latter inequality. We let $u = 1$, $v = \left|\frac{y}{x}\right|$ and $r = p\gamma$ and we get

$$\begin{aligned}
&|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \\
&\qquad \leq k_{1/p} k_{p\gamma} |y|^\gamma \mathbb{1}_{\{i(x)\leq|y|\}} + k_{1/p}|x|^\gamma \Big((1 + |\frac{y}{x}|)^{p\gamma} - 1 \Big) \Big(\frac{1}{|x|^{p\gamma}} + 1 \Big)^{\frac{1-p}{p}} \mathbb{1}_{\{|y|<i(x)\}}.
\end{aligned}$$

Since $|x| \geq 1$, $i(x)/|x|$ is bounded, and since $p > 1$, $(1/|x|^{p\gamma} + 1)^{(1-p)/p} \leq 1$. Using that $p\gamma > 2$ and the fact that $|y|/|x|$ is bounded, we obtain the existence of a $k' > 0$ such that

$$\Big((1 + \frac{|y|}{|x|})^{p\gamma} - 1 \Big) \leq k' \frac{|y|}{|x|}.$$

Taking $k = \max(k_{1/p} k_{p\gamma}, k_{1/p} k')$, we get the second inequality of the first part of Lemma 3.2.4.

We proceed with the second part and we note that, since $p\gamma > 2$, $h_{p,\gamma}$ is twice differentiable with

$$h''_{p,\gamma}(x) = \gamma\, |x|^{p\gamma-2} \big[(\gamma - 1)|x|^{p\gamma} + p\gamma - 1 \big] (1 + |x|^{p\gamma})^{1/p-2}.$$

Moreover, since $\gamma < \alpha < 2$, $h''_{p,\gamma} \in \mathrm{L}^\infty$. We split $(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x)$ into three terms

$$\begin{aligned}
\mathcal{A}_{\alpha,\beta} h_{p,\gamma}(x) = -\gamma \frac{|x|^{p\gamma+\beta-1}}{(1 + |x|^{p\gamma})^{1-1/p}} + \int_{|y|\leq 1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x) \Big] \nu(\mathrm{d}y) \\
+ \int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) \Big] \nu(\mathrm{d}y).
\end{aligned}$$

The first term on the right hand side is equivalent to $-\gamma|x|^{\gamma+\beta-1}$ at infinity, while for the second term, since $|y| \leq 1$, we have

$$\Big| h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x) \Big| \leq y^2 \sup_{|z|\leq 1} |h''_{p,\gamma}(x+z)| \leq y^2 \|h''_{p,\gamma}\|_\infty.$$

Hence

$$\Big| \int_{|y|\leq 1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x) \Big] \nu(\mathrm{d}y) \Big| \leq c_\alpha \|h''_{p,\gamma}\|_\infty,$$

where $c_\alpha := \int_{|y|\leq 1} y^2 \nu(\mathrm{d}y)$. We use the first part of the lemma to estimate the third term on the right hand side of the expression of $\mathcal{A}_{\alpha,\beta} h_{p,\gamma}(x)$. There are two situations: if $|x| \geq 1$, we get

$$\Big| h_{p,\gamma}(x+y) - h_{p,\gamma}(x) \Big| \leq k(|y||x|^{\gamma-1} \mathbb{1}_{\{|y|\leq i(x)\}} + |y|^\gamma \mathbb{1}_{\{i(x)<|y|\}}).$$

Hence

$$\begin{aligned}
&\Big| \int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) \Big] \nu(\mathrm{d}y) \Big| \\
&\qquad \leq k|x|^{\gamma-1} \int_{\{i(x)\geq|y|>1\}} |y| \nu(\mathrm{d}y) + k \int_{\{\max(1,i(x))\leq|y|\}} |y|^\gamma \nu(\mathrm{d}y) \\
&\qquad\qquad \leq k|x|^{\gamma-1} \int_{\{i(x)\geq|y|>1\}} |y| \nu(\mathrm{d}y) + k c'_{\alpha,\gamma}\,,
\end{aligned}$$

where $c'_{\alpha,\gamma} := \int_{\{|y|>1\}} |y|^\gamma \nu(\mathrm{d}y)$. Since $i(x) = O(|x|)$, as $|x| \to \infty$,

$$k|x|^{\gamma-1} \int_{\{i(x)\geq|y|>1\}} |y|\nu(\mathrm{d}y) = O(|x|^{\gamma-1}) + O(|x|^{\gamma-\alpha}), \qquad \text{as } |x| \to \infty.$$

If $|x| < 1$, since $|y| > 1$,

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq k|y|^\gamma,$$

so

$$\Big| \int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big| \leq \int_{|y|>1} |y|^\gamma \nu(\mathrm{d}y) < +\infty.$$

Denote by u the continuous function $-\mathcal{A}_{\alpha,\beta} h_{p,\gamma}$. Putting together the previous estimates we obtain that, since $\beta > 1$ and $\frac{2}{p} < \gamma < \alpha$,

$$u(x) \sim \gamma|x|^{\gamma+\beta-1}, \qquad \text{as } |x| \to \infty,$$

and since $\gamma > 2 - \beta$,

$$1 + |x| = o(u(x)), \qquad \text{as } |x| \to \infty.$$

Set $K = [k^-, k^+]$, with

$$k^+ := \inf\{x > 0 : y \geq x \Rightarrow u(y) > y+1\}, \;\; k^- := \sup\{x < 0 : y \leq x \Rightarrow u(y) > -y+1\}\,,$$

and

$$d := -\inf_{\{x\in K\}} (u(x) - 1 - |x|), \quad f_{p,\alpha,\beta,\gamma}(x) := u(x)\mathbb{1}_{K^c} + (1+|x|)\mathbb{1}_K\,.$$

Then relations (3.2.19) and (3.2.20) hold true and the proof is complete. $\square$

Proof of Proposition 3.2.3. By (3.0.7$_2$), we can write

$$\varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\frac{\mathcal{V}^\varepsilon_t}{\varepsilon^\theta}\Big) = \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) \tag{3.2.22}$$

and the first equality in (3.2.18) is clear. Since $2 - \beta < \frac{\alpha}{2}$ and $\beta > 1$, we can fix q such that $\frac{2}{p} \vee (2-\beta) < \gamma < 2\gamma < q\gamma < \alpha$ and $2 < q < \frac{\beta-1}{\alpha} + 2$. By noting that $h_{p,\gamma}(x)^q = h_{\frac{p}{q},q\gamma}(x)$, we can write

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} h_{\frac{p}{q},q\gamma}\Big(V_{t\,\varepsilon^{-\alpha\theta}}\Big)\Big)\Big].$$

Employing Itô's formula with $h_{\frac{p}{q},q\gamma}$, we get

$$h_{\frac{p}{q},q\gamma}(V_t) - h_{\frac{p}{q},q\gamma}(V_0) = R_t + \int_0^t (\mathcal{A}_{\alpha,\beta}\, h_{\frac{p}{q},q\gamma})(V_s)\mathrm{d}s, \tag{3.2.23}$$

where

$$R_t := \int_0^t \int_{\mathbb{R}} \Big(h_{\frac{p}{q}, q\gamma}(V_s + y) - h_{\frac{p}{q}, q\gamma}(V_s) \Big) \tilde{N}(\mathrm{d}y, \mathrm{d}s).$$

By Lemma 3.2.4 applied to the function $h_{\frac{p}{q}, q\gamma}$, we see that there exists $c > 0$ such that, for all $t \in [0, T]$,

$$\int_0^t (\mathcal{A}_{\alpha,\beta}\, h_{\frac{p}{q}, q\gamma})(V_s) \mathrm{d}s \le c\, t.$$

Moreover, let us note that $h_{\frac{p}{q}, q\gamma}$ is continuous and that $h_{\frac{p}{q}, q\gamma}(x) \sim |x|^{q\gamma}$, as $|x| \to \infty$. Hence, by the choice of q, we have $h_{\frac{p}{q}, q\gamma} \in \mathrm{L}^1(m_{\alpha,\beta})$. Taking the expectation in (3.2.23), we obtain

$$\begin{aligned} \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t \in [0,T]} h_{\frac{p}{q}, q\gamma} \Big(V_{t\, \varepsilon^{-\alpha\theta}} \Big) \Big) \Big] &\le \varepsilon^{q\frac{\alpha\theta}{2}}\, \| h_{\frac{p}{q}, q\gamma} \|_{\mathrm{L}^1(m_{\alpha,\beta})} + \varepsilon^{(q-2)\frac{\alpha\theta}{2}}\, cT \\ &\quad + \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big(\sup_{t \in [0,T]} R_{t\, \varepsilon^{-\alpha\theta}} \Big). \end{aligned}$$

Since $q > 2$, the first and the second term converge to 0. For the last term, we use the Kunita-Watanabe inequality (see for instance [1], p. 265). Since $V_0 \sim m_{\alpha,\beta}$, then for all t, $V_t \sim m_{\alpha,\beta}$ and there exists a positive constant C such that

$$\begin{aligned} \mathbb{E}\Big(\sup_{t \in [0,T]} R_{t\, \varepsilon^{-\alpha\theta}} \Big) &\le \mathbb{E}\Big(\sup_{t \in [0,T]} R^2_{t\, \varepsilon^{-\alpha\theta}} \Big)^{1/2} \\ &\le C \sqrt{T}\, \varepsilon^{-\frac{\alpha\theta}{2}} \iint_{\mathbb{R}^2} \big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x). \end{aligned}$$

It is sufficient to show that

$$\iint_{\mathbb{R}^2} \big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.2.24}$$

This fact is obtained by using Lemma 3.2.4. If $|x| \ge 1$,

$$\big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \big)^2 \le k^2 (|y|^2 |x|^{2q\gamma - 2} \mathbb{1}_{\{|y| \le i(x)\}} + |y|^{2q\gamma} \mathbb{1}_{\{i(x) < |y|\}}),$$

hence

$$\int_{\mathbb{R}} \Big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \Big)^2 \nu(\mathrm{d}y) = O(|x|^{2q\gamma - \alpha}), \text{ as } |x| \to +\infty,$$

and, since $q < \frac{\beta - 1}{\alpha} + 2$, we get (3.2.24). If $|x| < 1$,

$$\big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \big)^2 \le k^2 (|y|^2 \mathbb{1}_{\{|y| \le 1\}} + |y|^{2q\gamma} \mathbb{1}_{\{|y| > 1\}})$$

and $\int_{\mathbb{R}^2} \big(h_{\frac{p}{q}, q\gamma}(x+y) - h_{\frac{p}{q}, q\gamma}(x) \big)^2 \nu(\mathrm{d}y)$ is finite independently of x. Since $m_{\alpha,\beta}$ is a probability measure, (3.2.24) is verified again. □

3.2.2 Results on the position process

We are ready to give the proof of our main result concerning the behaviour of the position process. Recall that, thanks to Lemma 3.2.1, we assume that the initial distribution of V is the measure $\mathscr{m}_{\alpha,\beta}$.

Proof of Theorem 3.0.5 for the case $\alpha \in (0,2)$. Thanks to (3.2.19), Theorem 3.2, p. 924 in [17] applies and we deduce that the Poisson equation $\mathscr{A}_{\alpha,\beta} g = \mathrm{id}$ admits a solution $\mathscr{g}_{\alpha,\beta}$ satisfying $|\mathscr{g}_{\alpha,\beta}| \leq c(h_{p,\gamma}+1)$, with c a positive constant. Applying the Itô-Levy formula with $\mathscr{g}_{\alpha,\beta}$, we get

$$\mathscr{g}_{\alpha,\beta}(V_t) - \mathscr{g}_{\alpha,\beta}(V_0) = \int_0^t V_s \mathrm{d}s + M_t, \tag{3.2.25}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}} [\mathscr{g}_{\alpha,\beta}(z+V_s) - \mathscr{g}_{\alpha,\beta}(V_s)] \tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{3.2.26}$$

Step 1) We prove that M, given by the latter formula, is a square integrable martingale. On one hand we have

$$\mathbb{E}[\mathscr{g}_{\alpha,\beta}(V_t)^2] = \mathbb{E}[\mathscr{g}_{\alpha,\beta}(V_0)^2] = \int_{\mathbb{R}} \mathscr{g}_{\alpha,\beta}(x)^2 \mathscr{m}_{\alpha,\beta}(\mathrm{d}x) < \infty.$$

Indeed, recall that $h_{p,\gamma}^2$ is continuous and it behaves as $|x|^{2\gamma}$, as $|x| \to \infty$. Recalling that γ was chosen such that $\frac{4}{p} \vee (4-2\beta) < 2\gamma < \alpha$, by using (3.2.11), we see that

$$\int_{\mathbb{R}} h_{p,\gamma}(x)^2 \mathscr{m}_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.2.27}$$

We point out that the assumption $\beta + \frac{\alpha}{2} > 2$ is essential for the latter condition of integrability.

On the other hand, we can write

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^2\right] = \mathbb{E}\int_0^t \int_0^t V_u V_s \,\mathrm{d}u\mathrm{d}s = 2\mathbb{E}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, V_u V_s \\
\leq 2\mathbb{E}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, |V_u||V_s|.
\end{aligned}$$

Using the Markov property and that V_u and V_0 follow the invariant law, we get, for $u < s$, $\mathbb{E}\big(|V_s||V_u|\big) = \mathbb{E}\big(|V_{s-u}||V_0|\big)$. Therefore

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] \leq 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, \mathbb{E}\big(|V_{s-u}||V_0|\big) = 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, \mathbb{E}\big(|V_u||V_0|\big) \\
= 2\int_0^t \mathrm{d}s\, \mathbb{E}\Big(|V_0| \int_0^s \mathscr{T}_u |\mathrm{id}|(V_0) \mathrm{d}u\Big). \quad (3.2.28)
\end{aligned}$$

Applying Theorem 3.2, p. 924 in [17], we deduce that the Poisson equation $\mathcal{A}_{\alpha,\beta}\, g = |\mathrm{id}|$ admits a solution $\tilde{g}_{\alpha,\beta}$, This solution satisfies $|\tilde{g}_{\alpha,\beta}| \leq c'(h_{p,\gamma}+1)$, with c' a positive constant. Moreover

$$\int_0^s \mathcal{T}_u |\mathrm{id}|(V_0)\mathrm{d}u = \mathcal{T}_s \tilde{g}_{\alpha,\beta}(V_0) - \tilde{g}_{\alpha,\beta}(V_0).$$

By injecting this equality in (3.2.28), we get

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] \leq 2\int_0^t \mathbb{E}\left(|V_0||\mathcal{T}_s \tilde{g}_{\alpha,\beta}(V_0) - \tilde{g}_{\alpha,\beta}(V_0)|\right)\mathrm{d}s \\ = 2\int_0^t \mathrm{d}s \int_{\mathbb{R}} |x||\mathcal{T}_s \tilde{g}_{\alpha,\beta}(x) - \tilde{g}_{\alpha,\beta}(x)| m_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

At this level, we need to apply the Hölder inequality to conclude that

$$\mathbb{E}\Big[\int_0^t V_s \mathrm{d}s\Big]^2 < \infty. \tag{3.2.29}$$

Firstly, if $\beta < 2$ then we choose γ close enough to $2-\beta$ such that $\tilde{g}_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})$. Since $\frac{3-\beta}{2-\beta} > 1$, using the second part of Lemma 3.2.2, we get

$$\|\mathcal{T}_s \tilde{g}_{\alpha,\beta} - \tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})} \leq 2\|\tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})}.$$

By the Hölder inequality and the fact that $|\mathrm{id}| \in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, we get (3.2.29). Secondly, if $\beta \geq 2$, we choose $\gamma < 1$ close enough to 0 such that $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$. Since $\tilde{g}_{\alpha,\beta} \in \mathrm{L}^{1/\gamma}(m_{\alpha,\beta})$, using again Lemma 3.2.2, we get

$$\|\mathcal{T}_t \tilde{g}_{\alpha,\beta} - \tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})} \leq 2\|\tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})}.$$

Since $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, we can apply the Hölder inequality and get (3.2.29) again.

We conclude that M given by (3.2.26) is a square integrable martingale. Moreover, we can compute its quadratic variation

$$\langle M\rangle_t = \int_0^t \int_{\mathbb{R}} [g_{\alpha,\beta}(y+V_s) - g_{\alpha,\beta}(V_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s. \tag{3.2.30}$$

Hence

$$\mathbb{E}[\langle M\rangle_t] = t\iint_{\mathbb{R}^2} [g_{\alpha,\beta}(x+y) - g_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.2.31}$$

Let us remark here that for the step 1), we don't use that ℓ is symmetric, we only use the result 1.3.3.

Step 2) Performing a simple time change in (3.2.25), we see that the process in (3.0.10) can be written

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} \mathcal{X}_t^{\varepsilon} = \varepsilon^{\frac{\alpha\theta}{2}}\Big[g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big] - \varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}. \tag{3.2.32}$$

In this step, we show that the martingale term on the right hand side of the latter equality converges to a Brownian motion by using Whitt's theorem (see Theorem 2.1 (ii) in [45], pp. 270-271). We need to verify the hypotheses of this result. Indeed, since the function

$$x \mapsto \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) \in \mathrm{L}^1(\mathcal{m}_{\alpha,\beta}),$$

by using (3.2.30) and the ergodic theorem (3.2.12), we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\,\varepsilon^{-\alpha\theta}}\rangle_t &= \lim_{\varepsilon\to 0}\varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(y+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s \\ &= t \iint_{\mathbb{R}^2} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

The condition (6) in [45], p. 271 is fulfilled. Again by (3.2.30), we see that $\langle M\rangle$ has no jump, hence the condition (4) in [45], p. 270 is trivial. Let us note also that, by (3.2.25), the jumps of the martingale M_t are $J(M_t) := \mathcal{g}_{\alpha,\beta}(V_t) - \mathcal{g}_{\alpha,\beta}(V_{t-})$. Therefore we deduce that the jumps of the martingale term on the right hand side of (3.2.32) are

$$\begin{aligned}&J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big) := \varepsilon^{\frac{\alpha\theta}{2}} \Big[\mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t}\big) - \mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\Big] \\ &\leq c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t}\big)\big| + \big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\big| + 2\Big] \leq 2c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\sup_{t\in[0,T]}\Big|h_{p,\gamma}\Big(\varepsilon^{-\theta}V_t^{\varepsilon}\Big)\Big| + 1\Big],\end{aligned}$$

by using the fact that $|\mathcal{g}_{\alpha,\beta}| \leq c(h_{p,\gamma}+1)$ and (3.2.22). By Proposition 3.2.3,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{t\in[0,T]} J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big)^2\Big] = 0.$$

Therefore we can apply Whitt's theorem to deduce that $\big\{\varepsilon^{(\alpha\theta)/2} M_{t\,\varepsilon^{-\alpha\theta}} : t \geq 0\big\}$ converges in distribution (as a process) to $\kappa_{\alpha,\beta}^{1/2} B$, where B is the standard Brownian motion and

$$\kappa_{\alpha,\beta} := \iint_{\mathbb{R}^2} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{3.2.33}$$

Let us explain why the constant $\kappa_{\alpha,\beta}$ is positive. Indeed it suffices to note that ν is absolutely continuous with respect to the Lebesgue measure, that $\mathcal{m}_{\alpha,\beta}$ has a non-empty support, and that $\mathcal{g}_{\alpha,\beta}$ could not be a constant function, since $\mathcal{A}_{\alpha,\beta}\mathcal{g}_{\alpha,\beta} = \mathrm{id}$.

Step 3) By using that $|\mathcal{g}_{\alpha,\beta}| \leq c(h_{p,\gamma}+1)$, we get

$$\Big|\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathcal{g}_{\alpha,\beta}(V_0)\Big|^2 \leq 4c^2\Big(\Big|h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big|^2 + \Big|h_{p,\gamma}(V_0)\Big|^2 + 2\Big).$$

Hence, by using Proposition 3.2.3,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\varepsilon^{\alpha\theta}\sup_{t\in[0,T]}\Big|\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathcal{g}_{\alpha,\beta}(V_0)\Big|^2\Big] = 0.$$

Therefore, $\left\{\varepsilon^{(\alpha\theta)/2}\left[g_{\alpha,\beta}\left(V_{t\,\varepsilon^{-\alpha\theta}}\right) - g_{\alpha,\beta}(V_0)\right] : t \geq 0\right\}$ converges in probability to 0, uniformly on compact sets.

Step 4) Our processes are valued in the Skorokhod space of càdlàg functions $\mathrm{D}([0,\infty))$ endowed with J_1 Skorokhod topology (see [46], §3.3). It is not difficult to see that a sequence which converges in probability to 0, uniformly on compact sets, is also convergent in probability for J_1 metric, hence in distribution in J_1 topology. Recall that in the Skorokhod space, the summation is not a continuous map (see for instance [46], p. 84). In our case, the limits of the terms on the right hand side of equality (3.2.32) are, respectively, 0 and a Brownian motion, and have continuous paths. By using the joint convergence theorem (Theorem 11.4.5, p. 379 in [46]) and the continuous-mapping theorem (Theorem 3.4.3, p. 86 in [46]), we obtain the conclusion of Theorem 3.0.5. More precisely, the convergence in the theorem holds in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology. Let us note that our situation is simpler than in [19] since the limit is a continuous paths process. □

Proposition 3.2.5. *Assume that $\alpha \in (0,2)$ and $\beta + \frac{\alpha}{2} > 2$. The constant $\kappa_{\alpha,\beta}$ of the second part of Theorem* 3.0.5*, given in* (3.2.33)*, satisfies*

$$\kappa_{\alpha,\beta} = -2\int_{\mathbb{R}} x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{3.2.34}$$

Proof. Since, by (3.2.31) and (3.2.33), $\kappa_{\alpha,\beta} = \frac{1}{t}\mathbb{E}\left[M_t^2\right]$, for all $t > 0$, by taking $t = \varepsilon^{\alpha\theta}$ and using (3.2.25), we get

$$\begin{aligned}\kappa_{\alpha,\beta} = \varepsilon^{-\alpha\theta}\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0) - \int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big)^2\Big] &= \varepsilon^{-\alpha\theta}\Big\{\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\Big] \\ + \mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big)^2\Big] - 2\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big]\Big\}.\end{aligned} \tag{3.2.35}$$

The first term on the right hand side of (3.2.35) can be written :

$$\begin{aligned}\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\Big] &= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}})\Big] \\ &= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\mathbb{E}\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}})|V_0\Big)\Big] \\ &= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\big(\mathcal{T}_{\varepsilon^{\alpha\theta}}g_{\alpha,\beta}\big)(V_0)\Big] \\ &= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\Big(g_{\alpha,\beta}(V_0) + \int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(V_0)\mathrm{d}s\Big)\Big] \\ &= -2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(V_0)\mathrm{d}s\Big] = -2\int g_{\alpha,\beta}(x)\, m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(x)\mathrm{d}s \\ &= -2\,\varepsilon^{\alpha\theta}\int x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) - 2\int g_{\alpha,\beta}(x)\, m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}}\big((\mathcal{T}_s\mathrm{id}) - \mathrm{id}\big)(x)\mathrm{d}s.\end{aligned}$$

By using the Hölder inequality, we prove that,

$$\mathbb{E}\left[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\right] \sim -2\,\varepsilon^{\alpha\theta}\int x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x), \ \text{ as } \varepsilon \to 0. \tag{3.2.36}$$

We need to distinguish two cases following the position of β with respect to 2. Indeed, if $2 - \frac{\alpha}{2} < \beta < 2$,

$$g_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0} \|(\mathcal{T}_s \mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{3-\beta}(m_{\alpha,\beta})} = 0.$$

If $\beta \geq 2$,

$$g_{\alpha,\beta} \in \mathrm{L}^{\frac{1}{\gamma}}(m_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0} \|(\mathcal{T}_s \mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})} = 0.$$

By using (3.2.29) and Fubini's theorem, the second term on the right hand side of (3.2.35) can be written

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big)^2\Big] &= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big(V_s V_u\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big(V_{s-u} V_0\Big)\mathrm{d}u \\
&= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big(V_0\,(\mathcal{T}_{s-u}\mathrm{id})(V_0)\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\, \mathbb{E}\Big(V_0 \int_u^{\varepsilon^{\alpha\theta}} (\mathcal{T}_{s-u}\mathrm{id})(V_0)\,\mathrm{d}s\Big) \\
&= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\, \mathbb{E}\left[V_0\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}\big)(V_0) - g_{\alpha,\beta}(V_0)\Big)\right] \\
&= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u \int x\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}\big) - g_{\alpha,\beta}\Big)(x) m_{\alpha,\beta}(\mathrm{d}x).
\end{aligned}$$

Once again by the Hölder inequality, we prove that

$$\mathbb{E}\left[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big)^2\right] = o(\varepsilon^{\alpha\theta}), \ \text{ as } \varepsilon \to 0. \tag{3.2.37}$$

Indeed, if $2 - \frac{\alpha}{2} < \beta < 2$ then $\mathrm{id} \in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, and we note that

$$\lim_{\varepsilon\to 0} \sup_{0\leq u\leq \varepsilon^{\alpha\theta}} \|(\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}) - g_{\alpha,\beta}\|_{\mathrm{L}^{3-\beta/2-\beta}(m_{\alpha,\beta})} = 0.$$

Similarly, if $\beta \geq 2$ then $\mathrm{id} \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, and we see that

$$\lim_{\varepsilon\to 0} \sup_{0\leq u\leq \varepsilon^{\alpha\theta}} \|\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}) - g_{\alpha,\beta}\|_{\mathrm{L}^{\frac{1}{\gamma}}(m_{\alpha,\beta})} = 0.$$

Finally, the third term in (3.2.35) is analysed by using the Cauchy-Schwarz inequality and the behaviour of the other terms. We get that

$$-2\,\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big] = o(\varepsilon^{\alpha\theta}), \ \text{ as } \varepsilon \to 0. \tag{3.2.38}$$

Putting together (3.2.35)-(3.2.37), we obtain that

$$\kappa_{\alpha,\beta} = -2\int x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) + o(1), \ \text{ as } \varepsilon \to 0.$$

and the result is proved. $\square$

Chapter 4

Asymptotic stability for SDEs driven by non symmetric α-stable processes

4.1 Introduction

The first goal of this chapter is to extend Theorem 1.3.3 obtained by [38]. One considers the stochastic differential equation

$$\mathrm{d}X_t = -f(X_t)\mathrm{d}t + \mathrm{d}L_t \tag{4.1.1}$$

where f is a quickly increasing to infinity function and L is a symmetric α-stable Lévy motion and one studies the exact rate of decay of the tail probabilities of the random variables X_t, $t > 0$. In Remark 3.2, p. 76, of [38], the authors conjecture that their main result remains true without the assumption of symmetry of the Lévy process. The first part of this chapter contains a proof of this conjecture by trying to reduce the technical difficulties announced in the cited remark in [38] by assuming that the Lévy process is α-stable.

In the previous chapter, one studied the dynamics of a particle whose speed satisfies a one-dimensional stochastic differential equation driven by a small symmetric α-stable Lévy process in a potential of the form a power function of exponent $\beta+1$. A scaling limit of the position process with this speed was studied and it was proved that the limit in distribution is Brownian. More precisely, one considers the same stochastic differential equation as in Chapter 3

$$dv_t^\varepsilon = \varepsilon d\ell_t - \frac{1}{2}\mathcal{U}'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = 0, \tag{4.1.2}$$

with the potential $\mathcal{U}(x) := \frac{2}{\beta+1}|x|^{\beta+1}$ and assume that ℓ is a non-symmetric α-stable Lévy noise. By Proposition 1.1.10, $\alpha < 2$ and the Lévy measure of ℓ is

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\left[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\right]\mathrm{d}z.$$

If $\alpha = 1$, $a_+ = a_-$ and we will not consider this case so $\alpha \in (0,2) \setminus \{1\}$. We have $a_+ \neq a_-$ and we assume that $a_+ \neq 0$ and $a_- \neq 0$.

In the second part of the chapter, one explains what are the differences in order to obtain a similar result as in Chapter 3 when the driving noise for the speed is a non-symmetric α-stable Lévy process. To get our result, one uses the exact rate of decay of the tail probabilities for the speed process.

4.2 Tails of invariant measure of the solution

4.2.1 Notations and results

In this section, we denote by L a non-symmetric α-stable Lévy process having its Lévy measure given by

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z. \tag{4.2.3}$$

with $\alpha \in (0,2)\setminus\{1\}$ and $a_+ \neq a_-$. Moreover we will always suppose that $a_+ \neq 0$ and $a_- \neq 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $f(0) = 0$ and regularly varying at infinity with exponent $\beta > 1$: for all $a > 0$, $\lim_{x\to\infty} \frac{f(ax)}{f(x)} = a^\beta$. Recall that the process X satisfies

$$X_t := x + L_t - \int_0^t f(X_s)\mathrm{d}s,\ t \geq 0. \tag{4.2.4}$$

The statement of our main result is the following:

Theorem 4.2.1. *Denote, for all $u > 0$*

$$h(u) := \int_u^{+\infty} \frac{\nu((y,+\infty))}{f(y)}\mathrm{d}y\,. \tag{4.2.5}$$

Then

$$\lim_{u\to+\infty} \frac{\mathbb{P}_x(X_t > u)}{h(u)} = 1 \tag{4.2.6}$$

uniformly with respect to $x \in \mathbb{R}$ and $t \geq 1$.

As a consequence we obtain the behaviour of the tail for the invariant measure. According to Theorem 1.3.2, and under the assumptions on the function f, the exponential ergodicity of the solution X of (4.1.1) is insured. Moreover the unique invariant measure $m_{\alpha,\beta}$ satisfies

$$\forall x \in \mathbb{R}, \quad \|\mathbb{P}_x^t - m_{\alpha,\beta}\|_{\mathrm{TV}} = \mathrm{O}(\exp(-Ct)), \quad \text{as } t \to \infty, \tag{4.2.7}$$

where $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation and $\mathbb{P}_x^t$ is the distribution of X_t starting from x. Therefore letting t goes to infinity in Theorem (4.2.1), we get:

Corollary 4.2.2. *Under the same assumptions as in* Theorem 4.2.1*, we have*

$$\lim_{u\to+\infty} \frac{m_{\alpha,\beta}((u,+\infty))}{h(u)} = 1. \tag{4.2.8}$$

4.2.2 Proof of the main result

We split the proof of Theorem 4.2.1 in several steps.

Step 1. Introduce, for $\sigma > 0$ and some $c > 0$ to be chosen, the Lévy process $L^{(\sigma)}$ with the following small jumps prescribed by the Lévy measure

$$\nu^{(\sigma)}(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<-\sigma\}} + a_+\mathbb{1}_{\{z>c\sigma\}}\big]\,\mathrm{d}z. \tag{4.2.9}$$

The process $L^{(\sigma)}$ has a finite number of jumps on each finite interval of time. Denote by T_j the time when the j-th jump occurs (with the convention $T_0 = 0$) and by $W_j^{(\sigma)}$ its size. The random variables $(W_j^{(\sigma)})$ are i.i.d. We will choose the constant c such that, for all y and σ,

$$\mathbb{E}\big(W_1^{(\sigma)} \mathbb{1}_{\{-y \le W_1^{(\sigma)} \le cy\}}\big) = 0.$$

Since the probability density of $W_1^{(\sigma)}$ is given by

$$z \mapsto \frac{1}{\lambda_\sigma} |z|^{-1-\alpha} \big[a_- \mathbb{1}_{\{z<-\sigma\}} + a_+ \mathbb{1}_{\{z>c\sigma\}}\big] \quad \text{with} \quad \lambda_\sigma := \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}). \tag{4.2.10}$$

we deduce that

$$\mathbb{E}\Big(W_1^{(\sigma)} \mathbb{1}_{\{-y \le W_1^{(\sigma)} \le cy\}}\Big) = \frac{y^{1-\alpha} - \sigma^{1-\alpha}}{\lambda_\sigma (1-\alpha)} \big(a_+ c^{1-\alpha} - a_-\big) = 0.$$

Therefore the only possible value of the constant is

$$c = \Big(\frac{a_-}{a_+}\Big)^{1/(1-\alpha)}. \tag{4.2.11}$$

Let us point out that, for $u > c\sigma > 0$ by the definition of $\nu^{(\sigma)}$

$$\nu^{(\sigma)}\big((u, +\infty)\big) = \nu\big((u, +\infty)\big) =: \rho(u). \tag{4.2.12}$$

Step 2. Let us denote

$$X_t^{(\sigma)} := x + L_t^{(\sigma)} - \int_0^t f(X_s^{(\sigma)})\, \mathrm{d}s,\ t \ge 0. \tag{4.2.13}$$

According to Theorem 19.25 in [22], p. 385, $X^{(\sigma)}$ converges in distribution to X, as σ tends to 0. To get (4.2.6) it is enough to prove that there exists σ_0, such that,

$$\Big|\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{h(u)} - 1\Big| \le \mathrm{o}(1), \qquad \text{as } u \to +\infty, \tag{4.2.14}$$

uniformly in $x \in \mathbb{R}$, $\sigma \le \sigma_0$ and $t \ge 1$.

Step 3. The ordinary differential equation

$$x(t) = x - \int^t f(x(s))\mathrm{d}s, \quad t \ge 0 \tag{4.2.15}$$

has a unique solution. As in [38], p. 93, we introduce, for all $u > 0$

$$g(u) := \int_u^{+\infty} \frac{1}{f(y)} \mathrm{d}y\,. \tag{4.2.16}$$

This function is clearly finite, non-negative, continuous and strictly decreasing for large u. Let us fix $1 \le s \le t$. It is no difficult to see that the solution of (4.2.15) satisfies $g(x(t)) = g(x(s)) + t - s$ and in particular, for any $u > 0$, if $x(t) > u$ then $g(u) > g(x(t)) \geq t - s$. We deduce that the solution of (4.2.15) on $[t - g(u), t]$ will end up, at time t, not higher than u.

At this level let us recall an important result from [38] (see Lemma 5.1, p. 94). Let $A > 0$ and denote by y, the solution of the deterministic equation (4.2.15) on each interval of the form (S_{i-1}, S_i) with $0 = S_0 < \cdots < S_n < A$ but with jumps at time S_i of a size j_i. More precisely

$$y'(t) = -f(y(t)), \quad \text{on } (S_{i-1}, S_i) \text{ and } \quad y(S_i) = y(S_i^-) + j_i, \quad y(0) = x. \quad (4.2.17)$$

As previously, it is not difficult to see that y satisfies $g(y(A)) = g(y(S_n)) + A - S_n$ and in particular, for any $u > 0$, if $y(A) > u$, then $A - S_n \leq g(u)$. Moreover, one can compare the solution x of (4.2.15) with y:

$$-\max_{k=1,\dots,n} \Big(\sum_{i=k}^{n} j_i \Big)_- \leq y(A) - x(A) \leq \max_{k=1,\dots,n} \Big(\sum_{i=k}^{n} j_i \Big)_+.$$

In particular if we set, for $a > 0$, $N(a) = \sup\{i \leq n : j_i + \cdots + j_n > a\}$ (=0 if the set is empty), then

$$\text{for } t \in [S_{N(a)}, A] \text{ such that } y(t) \leq b, \text{ we have } y(A) \leq a + b. \quad (4.2.18)$$

Step 4. For $t \geq 1$, denote by $N_t^{(\sigma)}$ the number of jumps of $L^{(\sigma)}$ during the interval $[0, t]$ and define, for all $a < 0$ and $b > 0$,

$$M_1^{(\sigma)}(a, b) := \sup\{j \leq N_t^{(\sigma)} : W_j^{(\sigma)} \notin [a, b]\}, \quad \text{and } = 0 \text{ if the set is empty.} \quad (4.2.19)$$

To simplify notations we will denote by τ_1 the time

$$\tau_1 := T_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}.$$

We can write

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)} > u) = \mathbb{P}_x\left(X_t^{(\sigma)} > u,\, \tau_1 < t - g(\delta u)\right) + \mathbb{P}_x\left(X_t^{(\sigma)} > u,\, \tau_1 \in [t - g(\delta u), t]\right) \\ := p_1(u) + p_2(u). \quad (4.2.20)\end{aligned}$$

Let us fix $s \leq t$ and for $\varepsilon, \gamma, \delta > 0$ and $u > 0$, introduce the event

$$A_{\varepsilon,\gamma,\delta,u,s} := \left\{ \sup_{\substack{1 \leq i \leq N_t^{(\sigma)} \\ s - g(\delta u) \leq T_i \leq s}} \sum_{i \leq j \leq N_t^{(\sigma)}} W_j^{(\sigma)} \mathbb{1}_{\{-\varepsilon u \leq W_j^{(\sigma)} \leq c\varepsilon u\}} \geq \gamma u \right\}. \quad (4.2.21)$$

We can state the following lemma:

Lemma 4.2.3. *If $(1 \vee c)\varepsilon \leq \frac{\gamma}{4}$ then there exist $u_0(\varepsilon, \gamma, \delta)$, σ_0 and a positive constant $C(\beta, \gamma)$ such that, for all $u \geq u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leq \sigma_0$,*

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leq C(\varepsilon, \gamma) g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1 \vee c))}. \quad (4.2.22)$$

Remark 4.2.4. *Let us point out that all the constants here do not depend on t.*

We postpone the proof of Lemma 4.2.3 and we proceed with the proof of our main result.

Step 5. To begin with, we study the term p_1 in (4.2.20). We can write

$$p_1(u) \leq \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X_t^{(\sigma)} > u,\ \tau_1 < t - g(\delta u)\}). \tag{4.2.23}$$

By using (4.2.18) we get,

$$X_t^{(\sigma)} \leq \delta u + \gamma u \quad \text{on the event} \quad \left\{A^c_{\varepsilon,\gamma,\delta,u,t} \cap \tau_1 < t - g(\delta u)\right\}.$$

By choosing $\delta + \gamma \leq 1$, the second term on the right hand side of (4.2.23) is equal to 0. Furthermore, assuming that $(1 \vee c)\varepsilon \leq \frac{\gamma}{4}$, using Lemma (4.2.3), we see that there exist $u_0(\varepsilon,\gamma,\delta)$ and σ_0 such that, for all $u \geq u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leq \sigma_0$,

$$p_1(u) \leq \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leq C(\varepsilon,\gamma) g(\delta u)(\rho(u))^{\frac{\gamma}{4\varepsilon(1\vee c)}}.$$

We turn now to the term p_2 in (4.2.20). Let us introduce, for all $a < 0$ and $b > 0$,

$$M_2^{(\sigma)}(a,b) := \sup\{j < M_1^{(\sigma)}(a,b) : W_j^{(\sigma)} \notin [a,b]\}, \tag{4.2.24}$$

and, to simplify again, we denote

$$\tau_2 = T_{M_2^{(\sigma)}(-\varepsilon u, c\varepsilon u)}.$$

We can write

$$\begin{aligned} p_2(u) = \mathbb{P}_x\left(X_t^{(\sigma)} > u,\ \tau_1 \in [t - g(\delta u), t]\right) \\ \leq \mathbb{P}\left(t - \tau_1 \leq g(\delta u),\ \tau_1 - \tau_2 \leq g(\delta u)\right) \\ + \mathbb{P}_x\left(X_t^{(\sigma)} > u,\ t - \tau_1 \leq g(\delta u),\ \tau_1 - \tau_2 > g(\delta u)\right) \\ =: p_{21}(u) + p_{22}(u). \end{aligned} \tag{4.2.25}$$

Step 6. First, we estimate p_{21}. Recalling that I_u is the number of jumps of L^σ in the interval $[t - g(\delta u), t]$, we get

$$\mathbb{P}(\tau_1 \leq t - g(\delta u)) = \mathbb{P}\Big(\forall j \in [|1, I_u|], \quad -\varepsilon u \leq W_j^{(\sigma)} \leq c\varepsilon u\Big).$$

By using the independence between I_u and the $W_i^{(\sigma)}$, we deduce

$$\begin{aligned} \mathbb{P}(\tau_1 \leq t - g(\delta u)) &= e^{-\lambda_\sigma g(\delta u)} \sum_{n=0}^{+\infty} \frac{(\lambda_\sigma g(\delta u))^n}{n!} \mathbb{P}(-\varepsilon u \leq W_1^{(\sigma)} \leq c\varepsilon u)^n \\ &= e^{-\lambda_\sigma g(\delta u)}(1 - \mathbb{P}(-\varepsilon u \leq W_1^{(\sigma)} \leq c\varepsilon u)) = e^{-\lambda_\sigma g(\delta u)} \mathbb{P}(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]). \end{aligned}$$

Since

$$\mathbb{P}(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]) = \frac{c^{1-\alpha} + c^{-\alpha}}{\lambda_\sigma} \rho(\varepsilon u),$$

we get

$$\mathbb{P}(\tau_1 \le t - g(\delta u)) = e^{-(c^{1-\alpha}+c^{-\alpha})g(\delta u)\rho(\varepsilon u)}.$$

By using the fact that $t-\tau_1$ and $\tau_1-\tau_2$ are independent and have the same distribution, we obtain

$$p_{21}(u) = \left(1 - e^{-(c^{1-\alpha}+c^{-\alpha})g(\delta u)\rho(\varepsilon u)}\right)^2 \le (c^{1-\alpha}+c^{-\alpha})^2(\rho(\varepsilon u))^2 g(\delta u)^2.$$

To estimate p_{22}, we fix η that will be chosen later. We can write

$$\begin{aligned} p_{22}(u) \le \mathbb{P}_x\Big(X_t^{(\sigma)} > u,\, t-\tau_1 \le g(\delta u),\, X_{\tau_1^-}^{(\sigma)} \le \eta u\Big) \\ + \mathbb{P}_x\left(t-\tau_1 \le g(\delta u),\, X_{\tau_1^-}^{(\sigma)} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\right) \\ =: p_{221}(u) + p_{222}(u). \quad (4.2.26) \end{aligned}$$

Step 7. We begin with the study of p_{221}. We have

$$\begin{aligned} p_{221}(u) &\le \mathbb{P}(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x\left(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \left\{X_t^{(\sigma)} > u,\, t-\tau_1 \le g(\delta u),\, X_{\tau_1^-}^{(\sigma)} \le \eta u\right\}\right) \\ &:= \mathbb{P}(A_{\varepsilon,\gamma,\delta,u,t}) + p_{\text{main}}(u). \end{aligned}$$

By using Lemma (4.2.3), for all $u \ge u_0(\varepsilon,\gamma,\delta)$ and $\sigma \le \sigma_0$,

$$\mathbb{P}(A_{\varepsilon,\gamma,\delta,u,t}) \le C(\varepsilon,\gamma)g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}. \quad (4.2.27)$$

Furthermore, by the definition of g and (4.2.18), for all $u \ge u_0$, on the event

$$A^c_{\varepsilon,\gamma,\delta,u,t} \cap \left\{X_t^{(\sigma)} > u,\, t-\tau_1 \le g(\delta u),\, X_{\tau_1^-}^{(\sigma)} \le \eta u\right\},$$

the magnitude $W^\sigma_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ of the jump at time τ_1 should satisfy

$$t-\tau_1 + g(\eta u + W^\sigma_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}) \le g((1-\gamma)u).$$

Hence, since g is positive and decreasing, we get

$$t-\tau_1 \le g((1-\gamma)u) \quad \text{and} \quad W^\sigma_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} \ge g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u.$$

At this level, we need to assume that $(1\vee c)\varepsilon+\gamma+\eta < 1$. For all $s \in (0, g((1-\gamma)u))$,

$$\begin{aligned} \mathbb{P}\Big(W^\sigma_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} &\ge g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u)\Big) \\ &= \mathbb{P}(W_1^\sigma \ge g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u \mid W_1^\sigma \notin (-\varepsilon u, c\varepsilon u)) \\ &= \frac{\mathbb{P}(W_1^\sigma \ge g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u))}{\mathbb{P}(W_1^\sigma \notin (-\varepsilon u, c\varepsilon u))} \\ &= \frac{\rho(g^{-1}(g((1-\gamma)u) - s) - \eta u)}{(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)}. \end{aligned}$$

Since $t-\tau_1$ and $W^{\sigma}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ are independent and $t-\tau_1 \sim \mathcal{E}((c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u))$, we obtain

$$p_{\text{main}}(u) \leq \int_0^{g((1-\gamma)u)} (c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u) e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \\ \times \frac{\rho(g^{-1}(g((1-\gamma)u)-s)-\eta u))}{(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)} \mathrm{d}s.$$

By doing the change of variable $y = g^{-1}(g((1-\gamma)u)-s)$ and by using that

$$e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \leq 1,$$

we get

$$\begin{aligned} p_{\text{main}}(u) &\leq \int_0^{g((1-\gamma)u)} \ell(g^{-1}(g((1-\gamma)u)-s)-\eta u)) \\ &= \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y-\eta u)}{f(y)}\mathrm{d}y \leq \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y(1-\eta/(1-\gamma)))}{f(y)}\mathrm{d}y \\ &= \left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha} \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y)}{f(y)}\mathrm{d}y \\ &= \left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u). \end{aligned} \quad (4.2.28)$$

Putting together (4.2.27) and (4.2.28), we deduce, for all $u \geq u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leq \sigma_0$,

$$p_{221}(u) \leq \left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + C(\varepsilon,\gamma)g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}.$$

It remains to estimate p_{222}. Since $\tau_1-\tau_2$ and $t-\tau_1$ are independent, we get

$$p_{222}(u) = \mathbb{P}(t-\tau_1 \leq g(\delta u)) \times \mathbb{P}_x(X^{(\sigma)}_{\tau_1^-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)).$$

We can write

$$\mathbb{P}_x\left(X^{(\sigma)}_{\tau_1^-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\right) \leq \mathbb{P}(A_{\varepsilon,\gamma,\delta,u,\tau_1}) \\ + \mathbb{P}_x\left(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \left\{X^{(\sigma)}_{\tau_1^-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\right\}\right).$$

Choosing γ, δ and ε small enough, we can assume that $\delta+\gamma<\eta$ and so by the same argument as for p_1, we obtain

$$\mathbb{P}_x\left(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \left\{X^{(\sigma)}_{\tau_1^-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\right\}\right) = 0.$$

So, by using Lemma (4.2.3) and the distribution of $t-\tau_1$, we get, for all $u \geq u_0(\varepsilon,\delta,\gamma)$ and $\sigma \leq \sigma_0$,

$$p_{222}(u) \leq C(\varepsilon,\delta,\gamma,\eta)(\rho(u))^{\left(1+\frac{\gamma}{4(1\vee c)\varepsilon}\right)} g(u)^2.$$

Step 8. Finally, summarizing all the inequalities, for ε, γ, δ and η such that $\delta+\gamma<\eta<1$, $(1\vee c)\varepsilon<\frac{\gamma}{4}$ and $(1\vee c)\varepsilon+\gamma+\eta<1$, there exist $u_0(\varepsilon,\gamma,\delta,\eta)$ and σ_0 such that, for all $u\geq u_0(\varepsilon,\gamma,\delta,\eta)$ and $\sigma\leq\sigma_0$,

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)}>u)\leq\left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha}h((1-\gamma)u)+(c^{1-\alpha}+c^{-\alpha})^2(\rho(\varepsilon u))^2g(\delta u)^2\\ +C(\varepsilon,\gamma,\delta,\eta)g(u)(\rho(u))^{\frac{\gamma}{4(1\vee c)\varepsilon}}.\end{aligned}$$

Since h is regularly varying at infinity with exponent $1-\alpha-\beta$, g is regularly varying at infinity with exponent $1-\beta$ and $\rho(u)$ is regularly varying at infinity with exponent $-\alpha$, choosing ε, γ, δ and η small enough, we get that for all $\xi>0$, there exists $u_0(\xi)$ such that, for all $u\geq u_0(\xi)$, all $x\in\mathbb{R}$ and all $t\geq1$,

$$\frac{\mathbb{P}_x(X_t^{(\sigma)}>u)}{h(u)}\leq1+\xi,$$

hence we have established the upper bound of the main result.

Remark 4.2.5. *At this level we note that, if instead of a regular variation at infinity of the function f, we made only the assumption $f(x)\geq\hat{f}(x)$ for all $x\geq A$ for some function $\hat{f}$ which is regularly varying at infinity with exponent greater than one, we would still have the upper bound, for all $\xi>0$, there exists $u_0(\xi)$ such that, for all $u\geq u_0(\xi)$, all $x\in\mathbb{R}$ and all $t\geq1$,*

$$\frac{\mathbb{P}_x(X_t^{(\sigma)}>u)}{\hat{h}(u)}\leq1+\xi\quad\text{with}\quad\hat{h}(u)=\int_u^{+\infty}\frac{\nu((y,+\infty))}{\hat{f}(y)}\mathrm{d}y.$$

Step 9. We proceed with the proof of the lower bound. For all $\varepsilon<1$, $\delta<1$ and $\eta<1$, we get, by the strong Markov property and (4.2.18)

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)}>u)&\geq\mathbb{P}_x(X_t^{(\sigma)}>u,\,\tau_1\geq t-g(u(1+\delta)),\,X_{\tau_1^-}^{(\sigma)}\geq-\eta u)\\ &\geq\int_0^{g(u(1+\delta))}(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s}\mathbb{P}_x(X_{(t-s)^-}^{(\sigma)}\geq-\eta u)\\ &\quad\times\int_{c\varepsilon u}^{+\infty}\mathbb{P}_{y-\eta u}(X_s^{\varepsilon u}>u)\frac{\nu(\mathrm{d}y)}{(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)}\mathrm{d}s.\end{aligned}$$

Observe that $X^{(\sigma)}$ has, under $\mathbb{P}_x$, the same distribution as $-X^{(\sigma)}$ under the distribution $\mathbb{P}_{-x}$ but with a drift $\hat{f}(x)=-f(-x)$ and a Levy process where a_+ and a_- are inverted. So by using the hypothesis on f and Remark (4.2.5), we obtain that for all $u\geq u_0$, for all $\sigma\leq\sigma_0$, all $x\in\mathbb{R}$ and all $s<g(u(1+\delta))$,

$$\mathbb{P}_x(X_{(t-s)^-}^{(\sigma)}\geq-\eta u)\geq1-r(u),$$

where r is a function converging to zero. In the sequel, the function r can change from line to line.

Observe that, according to (4.2.18) (and as we have done for p_1), if

$$y\geq\eta u+g^{-1}(g(u(1+\delta))-s)\tag{4.2.29}$$

then, under the distribution $\mathbb{P}_{y-\eta u}$, the event $\{X_s^{\varepsilon u} > u\}$ contains, up to an event of probability zero, the event $A^c_{\varepsilon,\delta,1+\delta,u,t}$. Hence, for all s and y satisfying (4.2.29), we get

$$\mathbb{P}_{y-\eta u}(X_s^{\varepsilon u} > u) \geq 1 - \mathbb{P}(A_{\varepsilon,\delta,1+\delta,u,t}).$$

Therefore, by using Lemma (4.2.3), for all $\sigma \leq \sigma_0$ and $u \geq u_0(\varepsilon, \delta)$,

$$\mathbb{P}_{y-\eta u}(X_s^{\varepsilon u} > u) \geq 1 - r(u),$$

for all s and y satisfying (4.2.29), as long as ε is small relatively to δ. So, for all $\varepsilon < 1$, $\delta < 1$ and $\eta < 1$ such that ε is small relatively to δ, for all $\sigma \leq \sigma_0$ and all $u \geq u_0(\varepsilon, \delta)$,

$$\begin{aligned}
\mathbb{P}_x(X_t^{(\sigma)} > u) \geq & \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s}\mathbb{P}_x(X^{(\sigma)}_{(t-s)^-} \geq -\eta u) \\
& \times \int_{\eta u + g^{-1}(g(u(1+\delta))-s)}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{\varepsilon u} > u)\nu(\mathrm{d}y)\mathrm{d}s \\
\geq & (1-r(u))^2 \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s}\rho(\eta u + g^{-1}(g(u(1+\delta))-s))\mathrm{d}s \\
\geq & (1-r(u))^2 e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)g(u(1+\delta))} \int_{u(1+\delta)}^{+\infty} \frac{\rho(\eta u + y)}{f(y)}\mathrm{d}y \\
\geq & (1-r(u))^3 \int_{u(1+\delta)}^{+\infty} \frac{\rho(y(1+\eta/(1+\delta)))}{f(y)}\mathrm{d}y \\
= & (1-r(u))^3 \left(1 + \frac{\eta}{1+\delta}\right)^{-\alpha} h(u(1+\delta)).
\end{aligned}$$

We conclude that, for all $\xi > 0$, choosing η, ε and δ small enough, there exist $u_0(\xi)$ and $\sigma_0(\xi)$ such that

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \geq 1 - \xi,$$

for all $u \geq u_0(\xi)$, all $\sigma \leq \sigma_0(\xi)$, all $x \in \mathbb{R}$ and $t \geq 1$. $\square$

Proof of Lemma 4.2.3. For simplicity we will skip the index σ. Recall that we denote $\rho(u) = \nu\big((u, +\infty)\big)$. Set

$$q := \frac{a_-}{a_- + a_+ c^{-\alpha}}$$

and recall that

$$\lambda_\sigma = \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}).$$

Let us note that, for all ε, u and σ, 0 is a quantile of order q for the random variable $W_1 \mathbb{1}_{\{-\varepsilon u \leq W_1 \leq c\varepsilon u\}}$. Indeed, by using (4.2.10),

$$\begin{aligned}
\mathbb{P}_x(W_1 \mathbb{1}_{\{-\varepsilon u \leq W_1 \leq c\varepsilon u\}} < 0) = \mathbb{P}_x(-\varepsilon u \leq W_1 \leq -\sigma) &= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} - a_-(\varepsilon u)^{-\alpha}) \\
&= \frac{q}{\sigma^{-\alpha}}(\sigma^{-\alpha} - (\varepsilon u)^{-\alpha}) \leq q,
\end{aligned}$$

and

$$\mathbb{P}_x(W_1\mathbb{1}_{\{-\varepsilon u\le W_1\le c\varepsilon u\}}\le 0)=\mathbb{P}_x(W_1\le-\sigma)+\mathbb{P}_x(W_1>c\varepsilon u)$$
$$=\frac{1}{\lambda_\sigma\alpha}(a_-\sigma^{-\alpha}+a_+c^{-\alpha}(\varepsilon u)^{-\alpha})\ge\frac{a_-\sigma^{-\alpha}}{\lambda_\sigma\alpha}=q.$$

Denote by I_u the number of jumps of L^σ in $[s-g(\delta u),s]$. By using Theorem 2.1 p. 50 in [32], we get

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s})\le\frac{1}{q}\mathbb{P}_x\Big(\sum_{i=1}^{I_u}W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}\ge\gamma u\Big).$$

I_u is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the W_i, hence by conditionning, we obtain

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s})\le\frac{1}{q}\exp(-\lambda_\sigma g(\delta u))\sum_{n\ge1}\frac{\Big(\lambda_\sigma g(\delta u)\Big)^n}{n!}\mathbb{P}_x\Big(\sum_{i=1}^{n}W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}\ge\gamma u\Big).\tag{4.2.30}$$

Since $W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}$ are i.i.d. centered random variables, bounded by $(1\vee c)\varepsilon u$, we can use Theorem 1 in [33], p. 201. We get

$$\mathbb{P}_x\Big(\sum_{i=1}^{n}W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}\ge\gamma u\Big)\le\exp\left[-\frac{\gamma}{2\varepsilon(1\vee c)}\operatorname{arcsinh}\Big(\frac{\gamma u^2\varepsilon(1\vee c)}{n\mathbb{V}\mathrm{ar}\big(\mathrm{W_1}\mathbb{1}_{\{-\varepsilon u\le W_1\le c\varepsilon u\}}\big)}\Big)\right].$$

By using that $W_1\mathbb{1}_{\{-\varepsilon u\le W_1\le c\varepsilon u\}}$ is a centered random variable, we deduce

$$\mathbb{V}\mathrm{ar}\big(\mathrm{W_1}\mathbb{1}_{\{-\varepsilon u\le W_1\le c\varepsilon u\}}\big)=\mathbb{E}\big(\mathrm{W_1^2}\mathbb{1}_{\{-\varepsilon u\le W_1\le c\varepsilon u\}}\big)$$
$$=\frac{1}{\lambda_\sigma}\Big(\int_{-\varepsilon u}^{-\sigma}a_-|z|^{1-\alpha}\mathrm{d}z+\int_{c\sigma}^{c\varepsilon u}a_+z^{1-\alpha}\mathrm{d}z\Big)\le\frac{\alpha(c^{1-\alpha}+c^{2-\alpha})}{\lambda_\sigma(2-\alpha)}\varepsilon^{2-\alpha}u^2\rho(u).$$

Setting $\hat{C}:=\frac{(1\vee c)(2-\alpha)}{\alpha(c^{1-\alpha}+c^{2-\alpha})}$, we can write

$$\mathbb{P}_x\Big(\sum_{i=1}^{n}W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}\ge\gamma u\Big)\le\exp\left[-\frac{\gamma}{2\varepsilon(1\vee c)}\operatorname{arcsinh}\Big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\Big)\right].$$

Since $\operatorname{arcsinh}(x)\sim\log(x)$ when $x\to+\infty$, there exists $a>0$ such that for all $x\ge a$, $\operatorname{arcsinh}(x)\ge\frac{1}{2}\log(x)$. Therefore, if $n\le\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\,\rho(u)}$, we get

$$\mathbb{P}_x\Big(\sum_{i=1}^{n}W_i\mathbb{1}_{\{-\varepsilon u\le W_i\le c\varepsilon u\}}\ge\gamma u\Big)\le\exp\left[-\frac{\gamma}{4\varepsilon(1\vee c)}\log\Big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\Big)\right]$$
$$=\Big(\frac{n\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}.$$

By injecting this result in (4.2.30), we obtain

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leq \frac{1}{q}\left(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\right)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\Big(I_u^{\gamma/(4\varepsilon(1\vee c))}\Big) + \frac{1}{q}\mathbb{P}_x\left(I_u > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right). \tag{4.2.31}$$

It is no difficult to see that, if ξ is a Poisson distributed random variable, for all $p \geq 1$, there exists C_p such that

$$\mathbb{E}(\xi^p) \leq C_p\big(\mathbb{E}(\xi) + \mathbb{E}(\xi)^p\big).$$

Since $(1 \vee c)\varepsilon \leq \frac{\gamma}{4}$, we can apply this result to I_u and we deduce

$$\mathbb{E}\Big(I_u^{\gamma/(4\varepsilon(1\vee c))}\Big) \leq C'_{\varepsilon,\gamma}\Big(\lambda_\sigma g(\delta u) + (\lambda_\sigma g(\delta u))^{\gamma/(4\varepsilon(1\vee c))}\Big).$$

We obtain an estimate for the first term on the right hand side of (4.2.31): there exists $C(\varepsilon,\gamma)$ such that

$$\frac{1}{q}\left(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\right)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\Big(I_u^{\gamma/(4\varepsilon(1\vee c))}\Big) \leq C(\varepsilon,\gamma)g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}. \tag{4.2.32}$$

To study the second term on the right hand side of (4.2.31), we denote

$$\theta := \log\left(\frac{\varepsilon^{\alpha-1}\gamma}{g(\delta u)\rho(u)}\right).$$

There exists $u_0(\varepsilon,\gamma,\delta)$ such that for all $u \geq u_0(\varepsilon,\gamma,\delta)$, θ is strictly positive. We get, for all $u \geq u_0(\varepsilon,\gamma,\delta)$

$$\begin{aligned}\mathbb{P}_x\left(I_u > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right) &= \mathbb{P}_x\left(e^{\theta I_u} > \exp\left(\theta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right)\right)\\ &\leq \exp\left((e^\theta - 1)\lambda_\sigma g(\delta u) - \theta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right),\end{aligned}$$

by using Markov's inequality. By using the expression of θ and choosing $C(\varepsilon,\gamma)$ and $u_0(\varepsilon,\gamma,\delta)$ large enough, we obtain

$$\mathbb{P}_x\left(I_u > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right) \leq C(\varepsilon,\gamma)\left(g(\delta u)\rho(u)\right)^{C(\varepsilon,\gamma)\lambda_\sigma/\rho(u)}. \tag{4.2.33}$$

Replacing (4.2.32) and (4.2.33) in (4.2.31), we get (4.2.22). □

4.3 Proof of Gaussian Asymptotic stability for SDEs driven by non symmetric α-stable processes

Recall that we consider the equation

$$\begin{cases} v_t^\varepsilon = \varepsilon\ell_t - \int_0^t \operatorname{sgn}(v_s^\varepsilon)|v_s^\varepsilon|^\beta \mathrm{d}s \\ x_t^\varepsilon = \int_0^t v_s^\varepsilon \mathrm{d}s, \end{cases}$$

where ℓ is an α-stable Lévy process with $\alpha \in]0;2[\backslash\{1\}$. It is a pure jump process with càdlàg paths and the Lévy measure is given by :

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z.$$

Denote

$$\mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad \mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon$$

satisfying

$$\mathcal{X}_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t \mathcal{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \mathcal{V}_t^\varepsilon = \mathcal{L}_t^\varepsilon - \frac{1}{\varepsilon^\alpha}\int_0^t \operatorname{sgn}(\mathcal{V}_s^\varepsilon)|\mathcal{V}_s^\varepsilon|^\beta \mathrm{d}s.$$

To simplify, denote

$$\theta = \theta_{\alpha,\beta} := \frac{\alpha}{\alpha+\beta-1} > 0, \quad \text{ if } \alpha+\beta-1>0.$$

Finally, we introduce

$$L_t^\varepsilon := \frac{\mathcal{L}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta} = \frac{\ell_{t\,\varepsilon^{-(\beta-1)\theta}}}{\varepsilon^{(\beta-1)\theta/\alpha}} \quad \text{et} \quad V_t^\varepsilon := \frac{\mathcal{V}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta}. \tag{4.3.1}$$

Since ℓ is self similar, L^ε is an α-stable Lévy process with the same Lévy measure and we have

$$\mathcal{X}_t^\varepsilon = \varepsilon^{(2-\beta)\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s^\varepsilon \mathrm{d}s \quad \text{and} \quad V_t^\varepsilon = L_t^\varepsilon - \int_0^t \operatorname{sgn}(V_s^\varepsilon)|V_s^\varepsilon|^\beta \mathrm{d}s.$$

We will skip the index ε in L and V for the rest of the proof.

Theorem 4.3.1 (The non symmetric case)**.** *Assume that $\beta + \frac{\alpha}{2} > 2$ and recall that $\theta \in (0,1)$. Then there exists a positive constant $\kappa_{\alpha,\beta}$ such that the process*

$$\Big\{\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\Big(x_{\varepsilon^{-\alpha}t}^\varepsilon - \varepsilon^{\theta-\alpha}t\int x m_{\alpha,\beta}(\mathrm{d}x)\Big) : t \geq 0\Big\} \tag{4.3.2}$$

converges in distribution toward a Brownian motion with diffusion coefficient $\kappa_{\alpha,\beta}$, when $\varepsilon \to 0$. Here $m_{\alpha,\beta}$ is the invariant measure of V. The constant $\kappa_{\alpha,\beta}$ has an integral representation.

4.3.1 The speed process

The drift still satisfies the condition of the Proposition 1.2.10 so there exists a global pathwise unique strong solution V for equation $(4.3.1_2)$.

The ergodic feature of the process V is a consequence of Theorem 1.3.2. Indeed, provided that $\beta > 1$, the drift coefficient $-f(x) := -\operatorname{sgn}(x)|x|^\beta$ and the jump measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z$ clearly satisfy the conditions in the cited result. Hence V is an exponential ergodic and Harris recurrent process having an unique invariant distribution, denoted, as in Chapter 3, by $m_{\alpha,\beta}$. Using Corollary 4.2.2 with f, $m_{\alpha,\beta}$ satisfies

$$m_{\alpha,\beta}((x,+\infty)) \underset{x\to+\infty}{\sim} \left(\frac{\theta a_+}{\alpha^2 x^{\alpha+\beta-1}}\right) \quad \text{and} \quad m_{\alpha,\beta}((-\infty,-x)) \underset{x\to+\infty}{\sim} \left(\frac{\theta a_-}{\alpha^2 x^{\alpha+\beta-1}}\right). \tag{4.3.3}$$

Clearly, under the hypothesis of Theorem 4.3.1 and $\beta+\frac{\alpha}{2}-2>0$, the identity function belongs to $\mathrm{id}\in \mathrm{L}^1(m_{\alpha,\beta})$. By the classical ergodic theorem, for all $f\in \mathrm{L}^1(m_{\alpha,\beta})$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(V_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) m_{\alpha,\beta}(\mathrm{d}x), \text{ a.s.} \tag{4.3.4}$$

so

$$\lim_{\varepsilon\to 0}\varepsilon^{\alpha\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s\mathrm{d}s = t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x).$$

Recall that we are interested on the behaviour as $\varepsilon\to 0$ of

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\Big(x^{\varepsilon}_{\varepsilon^{-\alpha}t} - \varepsilon^{\theta-\alpha}t\int x m_{\alpha,\beta}(\mathrm{d}x)\Big) = \varepsilon^{-\frac{\alpha\theta}{2}}\Big(\varepsilon^{\alpha\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s\mathrm{d}s - t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big). \tag{4.3.5}$$

In other words, we are studying a large time behaviour of a functional of V, hence it is quite natural to perform the study in steady state as in Chapter 3. This fact is contained in the following lemma (see also [2], Theorem 2.6, p. 194) which proof follows the same idea as in Chapter 3:

Lemma 4.3.2. *Suppose that $\beta+\frac{\alpha}{2}-2>0$. Assume that the process*

$$\left\{\varepsilon^{-\frac{\alpha\theta}{2}}\Big(\varepsilon^{\alpha\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s\mathrm{d}s - t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big) : t\geq 0\right\}$$

converges, as $\varepsilon\to 0$, in distribution toward a Brownian motion, provided that V is starting with $m_{\alpha,\beta}$ as an initial distribution. Then the same process converges in distribution toward a Brownian motion when $V_0=0$.

Proof. In this proof we will denote the process in (4.3.5) by $Z_{\varepsilon,0}(t)$, and for $\Delta\geq 0$,

$$Z_{\varepsilon,\Delta}(t) := \varepsilon^{-\frac{\alpha\theta}{2}}\Big(\varepsilon^{\alpha\theta}\int_{\Delta}^{t\varepsilon^{-\alpha\theta}+\Delta} V_s\mathrm{d}s - t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big).$$

First, let us prove that $Z_{\varepsilon,\Delta}(\cdot)$ converges in distribution, as $\Delta\to\infty$ and $\varepsilon\to 0$, toward a Brownian motion, when $V_0=0$. Denoting by μ_Δ the distribution of V_Δ, for each bounded continuous real function ψ on $\mathrm{C}([0,+\infty))$, by the Markov property, we have

$$\mathbb{E}_0\big[\psi(Z_{\varepsilon,\Delta}(\cdot))\big] = \mathbb{E}_{\mu_\Delta}\big[\psi(Z_{\varepsilon,0}(\cdot))\big].$$

We can write, for all $\varepsilon>0$,

$$\begin{aligned}\big|\mathbb{E}_{\mu_\Delta}\big[\psi(Z_{\varepsilon,0}(\cdot))\big] - \mathbb{E}_{m_{\alpha,\beta}}\big[\psi(Z_{\varepsilon,0}(\cdot))\big]\big| &\\ = \Big|\int_{\mathbb{R}}\mathbb{E}_y\big[\psi(Z_{\varepsilon,0}(\cdot))\big]\big(\mu_\Delta(\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big)\Big| &\\ \leq ||\psi||_\infty\int_{\mathbb{R}}\big|p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big| \leq ||\psi||_\infty||p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)||_{\mathrm{TV}}\,,&\end{aligned}$$

where $p(t,x,\mathrm{d}y) = \mathbb{P}_x(V_t \in \mathrm{d}y)$ is the transition kernel of V (and therefore we have $p(\Delta,0,\mathrm{d}y) = \mu_\Delta(\mathrm{d}y)$) , and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Since V is exponentially ergodic, we get that

$$\lim_{\Delta\to\infty} \left|\mathbb{E}_{\mu_\Delta}\left[\psi(Z_{\varepsilon,0}(\cdot))\right] - \mathbb{E}_{m_{\alpha,\beta}}\left[\psi(Z_{\varepsilon,0}(\cdot))\right]\right| = 0, \qquad \text{uniformly in } \varepsilon.$$

Second, by choosing $\Delta = \Delta(\varepsilon) = \varepsilon^{-\alpha\theta/4}$ we obtain

$$\begin{aligned}\sup_{t\geq 0}\left\{\left|Z_{\varepsilon,\Delta(\varepsilon)}(t) - \varepsilon^{-\frac{\alpha\theta}{2}}\left(\varepsilon^{\alpha\theta}\int_0^{t\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s \mathrm{d}s - t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\right)\right|\right\} &\leq \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s \\ &= \varepsilon^{\frac{\alpha\theta}{4}}\frac{1}{\Delta(\varepsilon)}\int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s.\end{aligned}$$

The right hand side term of the latter inequality tends to 0 almost surely, by using the ergodicity (4.3.4). Therefore

$$\varepsilon^{-\alpha\theta/2}\Big(\varepsilon^{\alpha\theta}\int_0^{\bullet\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s \mathrm{d}s - \bullet\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big)$$

converges in distribution, as $\varepsilon \to 0$, toward a Brownian motion when $V_0 = 0$. Clearly, we see that $\lim_{\varepsilon\to 0}(t - \Delta(\varepsilon)\varepsilon^{\alpha\theta}) = t$, and by applying a consequence of the continuous mapping theorem for the composition function stated in Lemma p. 151 in [6], we have that

$$\varepsilon^{-\alpha\theta/2}\Big(\varepsilon^{\alpha\theta}\int_0^{\bullet\varepsilon^{-\alpha\theta}} V_s \mathrm{d}s - (\bullet - \Delta(\varepsilon)\varepsilon^{\alpha\theta})\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big)$$

converges in distribution, as $\varepsilon \to 0$, toward a Brownian motion when $V_0 = 0$. Since $\Delta(\varepsilon)\varepsilon^{\frac{\alpha\theta}{2}}\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)$ converges, as $\varepsilon \to 0$, toward 0, we can conclude. □

In the sequel, we will always assume that V is starting with $m_{\alpha,\beta}$ as an initial distribution. Let us recall that there exists $b \in \mathbb{R}$ such that the infinitesimal generator of V is given by

$$(\mathcal{A}_{\alpha,\beta}\, g)(x) = (-\mathrm{sgn}(x)|x|^\beta + b)g'(x) + \int_{\mathbb{R}}\Big[g(x+y) - g(x) - yg'(x)\mathbb{1}_{|y|\leq 1}\Big]\nu(\mathrm{d}y)\,, \tag{4.3.6}$$

with the domain $D_{\mathcal{A}_{\alpha,\beta}}$. Also denote $(\mathcal{T}_t)_{t\geq 0}$ the semi-group associated to the operator $\mathcal{A}_{\alpha,\beta}$ or to the process V. The following lemma contains some useful properties of the process V.

Lemma 4.3.3.

1. *The domain $D_{\mathcal{A}_{\alpha,\beta}}$ contains the space of bounded twice differentiable functions $\mathrm{C}_b^2(\mathbb{R})$.*
2. *For all $p \geq 1$, $\mathcal{T}_t$ is a contraction semi-group on $\mathrm{L}^p(m_{\alpha,\beta})$ and for each $f \in \mathrm{L}^p(m_{\alpha,\beta})$,*

$$\lim_{t\to 0}\|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} = 0. \tag{4.3.7}$$

Proof. The proof is exactly the same as in Chapter 3, Lemma (3.2.2) by replacing the drift by $-\mathrm{sgn}(x)|x|^\beta + b$. □

Convergence in probability

The main result of this section concerns the behaviour of the speed process which is the same as in Chapter 3 and is described by using the same Lyapunov function.

Proposition 4.3.4. *Suppose that $\beta + \frac{\alpha}{2} > 2$ and let p and γ such that*

$$p > 1, \;\; p\gamma > 2, \;\; 2 - \beta < \gamma < \frac{\alpha}{2}. \tag{4.3.8}$$

Introduce the Lyapunov function

$$h_{p,\gamma}(x) := (1 + |x|^{p\gamma})^{1/p}. \tag{4.3.9}$$

Then, as $\varepsilon \to 0$, $\{\varepsilon^{\alpha\theta/2} h_{p,\gamma}(\varepsilon^{-\theta}\mathcal{V}_t^\varepsilon) : t \geq 0\}$ converges to 0 in probability uniformly on each compact time interval. More precisely, there exists $q > 2$ such that, for any fixed $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\varepsilon^{-\theta}\mathcal{V}_t^\varepsilon\Big)\Big)^q\Big] = \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = 0. \tag{4.3.10}$$

Let us recall that, by $(4.3.1_2)$, we can write

$$\varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\frac{\mathcal{V}_t^\varepsilon}{\varepsilon^\theta}\Big) = \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big). \tag{4.3.11}$$

The proof is exactly the same as in Chapter 3 by using Lemma (4.3.3) instead of Lemma (3.2.2) and it is essentially based on the fact that $h_{p,\gamma}$ is a Lyapunov function. Assume that $p\gamma > 2$ and $2 - \beta < \gamma < \alpha$. There exist a continuous function $f_{p,\alpha,\beta,\gamma}$, a compact set K and a constant d, depending only on p, α, β, γ, such that

$$\forall x \in \mathbb{R}, \;\; f_{p,\alpha,\beta,\gamma}(x) \geq 1 + |x|, \quad f_{p,\alpha,\beta,\gamma}(x) \underset{|x|\to\infty}{\sim} \gamma|x|^{\gamma+\beta-1}, \tag{4.3.12}$$

and

$$(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x) \leq -f_{p,\alpha,\beta,\gamma}(x) + d\,\mathbb{1}_K(x). \tag{4.3.13}$$

4.3.2 The position process $\mathcal{X}^\varepsilon$

We are ready to prove our main result concerning the behaviour of the position process. Recall that, thanks to Lemma 4.3.2, we assume that V is starting with $m_{\alpha,\beta}$ as an initial distribution. The proof follows the same ideas as in Chapter 3 but, since the Poisson's equation changes, new terms will appear in the computations.

Proof of Theorem 4.3.1. Thanks to (4.3.12), Theorem 3.2, p. 924 in [17] applies and we deduce that the Poisson equation $\mathcal{A}_{\alpha,\beta} g = \mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x)$ admits a solution $g_{\alpha,\beta}$ satisfying $|g_{\alpha,\beta}| \leq c(h_{p,\gamma}+1)$, with c a positive constant. Applying Itô-Levy's formula with $g_{\alpha,\beta}$, we get

$$g_{\alpha,\beta}(V_t) - g_{\alpha,\beta}(V_0) = \int_0^t V_s \mathrm{d}s - t \int x m_{\alpha,\beta}(\mathrm{d}x) + M_t, \tag{4.3.14}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}} [\mathscr{g}_{\alpha,\beta}(z+V_s) - \mathscr{g}_{\alpha,\beta}(V_s)]\tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{4.3.15}$$

Step 1) We prove that M given by the latter formula is a square integrable true martingale. On one hand we have

$$\mathbb{E}[\mathscr{g}_{\alpha,\beta}(V_t)^2] = \mathbb{E}[\mathscr{g}_{\alpha,\beta}(V_0)^2] = \int_{\mathbb{R}} \mathscr{g}_{\alpha,\beta}(x)^2 \mathscr{m}_{\alpha,\beta}(\mathrm{d}x) < \infty.$$

Indeed, recall that $h^2_{p,\gamma}$ is continuous and it behaves as $|x|^{2\gamma}$ in the neighbourhood of the infinity. Recalling that γ was chosen such that $\frac{4}{p} \vee (4-2\beta) < 2\gamma < \alpha$, by using (4.3.3), we see that

$$\int_{\mathbb{R}} h_{p,\gamma}(x)^2 \mathscr{m}_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{4.3.16}$$

We point out that the assumption $\beta + \frac{\alpha}{2} > 2$ is essential for the latter condition of integrability.

On the other hand, we can write

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^2\right] &= \mathbb{E}\int_0^t \int_0^t V_u V_s \,\mathrm{d}u\mathrm{d}s = 2\mathbb{E}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \, V_u V_s \\ &\leq 2\mathbb{E}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \, |V_u||V_s|.\end{aligned}$$

Using Markov's property and the fact that V_u and V_0 follow the invariant law, we get, for $u < s$, $\mathbb{E}\big(|V_s||V_u|\big) = \mathbb{E}\big(|V_{s-u}||V_0|\big)$. Therefore

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] &\leq 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \, \mathbb{E}\big(|V_{s-u}||V_0|\big) = 2\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \, \mathbb{E}\big(|V_u||V_0|\big) \\ &= 2\int_0^t \mathrm{d}s \, \mathbb{E}\Big(|V_0| \int_0^s \mathcal{T}_u |\mathrm{id}|(V_0)\mathrm{d}u\Big).\end{aligned}$$

Applying again Theorem 3.2, p. 924 in [17], the Poisson equation $\mathscr{A}_{\alpha,\beta}\, g = |\mathrm{id}|$ admits a solution $\tilde{\mathscr{g}}_{\alpha,\beta}$ satisfying $|\tilde{\mathscr{g}}_{\alpha,\beta}| \leq c'(h_{p,\gamma}+1)$ with c' a positive constant. Moreover

$$\int_0^s \mathcal{T}_u |\mathrm{id}|(V_0)\mathrm{d}u = \mathcal{T}_s \tilde{\mathscr{g}}_{\alpha,\beta}(V_0) - \tilde{\mathscr{g}}_{\alpha,\beta}(V_0).$$

Replacing in the latter inequality

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] &\leq 2\int_0^t \mathbb{E}\left(|V_0||\mathcal{T}_s \tilde{\mathscr{g}}_{\alpha,\beta}(V_0) - \tilde{\mathscr{g}}_{\alpha,\beta}(V_0)|\right)\mathrm{d}s \\ &= 2\int_0^t \mathrm{d}s \int_{\mathbb{R}} |x||\mathcal{T}_s \tilde{\mathscr{g}}_{\alpha,\beta}(x) - \tilde{\mathscr{g}}_{\alpha,\beta}(x)| \mathscr{m}_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

At this level, we need to apply the Hölder inequality to conclude that

$$\mathbb{E}\Big[\int_0^t V_s \mathrm{d}s\Big]^2 < \infty. \tag{4.3.17}$$

First, if $\beta < 2$ then we choose γ close enough to $2-\beta$ such that $\tilde{\mathcal{g}}_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(\mathcal{m}_{\alpha,\beta})$. Since $\frac{3-\beta}{2-\beta} > 1$, using the second part of Lemma 4.3.3, we get

$$\|\mathcal{T}_s\tilde{\mathcal{g}}_{\alpha,\beta} - \tilde{\mathcal{g}}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(\mathcal{m}_{\alpha,\beta})} \leq 2\|\tilde{\mathcal{g}}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(\mathcal{m}_{\alpha,\beta})}.$$

By the Hölder inequality and the fact that $|\mathrm{id}| \in \mathrm{L}^{3-\beta}(\mathcal{m}_{\alpha,\beta})$, we get (4.3.17). Second, if $\beta \geq 2$, we choose $\gamma < 1$ close enough to 0 such that $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(\mathcal{m}_{\alpha,\beta})$. Since $\tilde{\mathcal{g}}_{\alpha,\beta} \in \mathrm{L}^{1/\gamma}(\mathcal{m}_{\alpha,\beta})$, using again Lemma 4.3.3, we get

$$\|\mathcal{T}_t\tilde{\mathcal{g}}_{\alpha,\beta} - \tilde{\mathcal{g}}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(\mathcal{m}_{\alpha,\beta})} \leq 2\|\tilde{\mathcal{g}}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(\mathcal{m}_{\alpha,\beta})}.$$

Since $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(\mathcal{m}_{\alpha,\beta})$, we can apply the Hölder inequality and get (4.3.17) again.

We conclude that M given by (4.3.15) is a square integrable true martingale. Moreover, we can compute its quadratic variation

$$\langle M\rangle_t = \int_0^t \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(y+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s, \tag{4.3.18}$$

hence

$$\mathbb{E}[\langle M\rangle_t] = t\iint_{\mathbb{R}^2} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{4.3.19}$$

Step 2) Performing a simple time change in (4.3.14), we see that the process in (4.3.2) can be written

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\Big(\mathcal{X}_t^\varepsilon - \varepsilon^{\theta-\alpha}t\int x\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\Big) = \varepsilon^{\frac{\alpha\theta}{2}}\Big[\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathcal{g}_{\alpha,\beta}(V_0)\Big] - \varepsilon^{\frac{\alpha\theta}{2}}M_{t\,\varepsilon^{-\alpha\theta}}. \tag{4.3.20}$$

In this step, we show that the martingale term on the right hand side of the latter equality converges to a Brownian motion by using Whitt's theorem (see Theorem 2.1 (ii) in [45], pp. 270-271). We need to verify the hypotheses of this result. Indeed, since the function

$$x \mapsto \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) \in \mathrm{L}^1(\mathcal{m}_{\alpha,\beta}),$$

by using (4.3.18) and the ergodic theorem (4.3.4), we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\,\varepsilon^{-\alpha\theta}}\rangle_t = \lim_{\varepsilon\to 0}\varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} & \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(y+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)]^2\nu(\mathrm{d}y)\mathrm{d}s \\ &= t\iint_{\mathbb{R}^2} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2\nu(\mathrm{d}y)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

The condition (6) in [45], p. 271 is fulfilled. Again by (4.3.18), we see that $\langle M\rangle$ has no jump, hence the condition (4) in [45], p. 270 is trivial. Let us note also that, by (4.3.14), the jumps of the martingale M_t are $J(M_t) := \mathcal{g}_{\alpha,\beta}(V_t) - \mathcal{g}_{\alpha,\beta}(V_{t-})$.

Therefore we deduce that the jumps of the martingale term on the right hand side of (4.3.20) are

$$J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big) := \varepsilon^{\frac{\alpha\theta}{2}}\Big[g_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t}\big) - g_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\Big]$$
$$\le c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t}\big)\big| + \big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\big| + 2\Big] \le 2c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\sup_{t\in[0,T]}\Big|h_{p,\gamma}\Big(\varepsilon^{-\theta}V^{\varepsilon}_t\Big)\Big| + 1\Big],$$

by using the fact that $|g_{\alpha,\beta}| \le c(h_{p,\gamma}+1)$ and (4.3.11). By Proposition 4.3.4,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{t\in[0,T]} J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big)^2\Big] = 0.$$

Therefore we can apply Whitt's theorem to deduce that $\big\{\varepsilon^{(\alpha\theta)/2} M_{t\,\varepsilon^{-\alpha\theta}} : t \ge 0\big\}$ converges in distribution (as a process) toward $\kappa_{\alpha,\beta}^{1/2} B$, where B is a standard Brownian motion and

$$\kappa_{\alpha,\beta} := \iint_{\mathbb{R}^2} [g_{\alpha,\beta}(x+y) - g_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{4.3.21}$$

The constant $\kappa_{\alpha,\beta}$ is positive by noting that ν is absolutely continuous with respect to the Lebesgue measure, that $m_{\alpha,\beta}$ has a non-empty support, and that $g_{\alpha,\beta}$ could not be a constant function, since $\mathcal{A}_{\alpha,\beta} g_{\alpha,\beta} = \mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x)$.

Step 3) By using that $|g_{\alpha,\beta}| \le c(h_{p,\gamma}+1)$, we get

$$\Big|g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big|^2 \le 4c^2\Big(\Big|h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big|^2 + \Big|h_{p,\gamma}(V_0)\Big|^2 + 2\Big)$$

hence, using Proposition 4.3.4,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\varepsilon^{\alpha\theta}\sup_{t\in[0,T]}\Big|g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big|^2\Big] = 0$$

hence $\big\{\varepsilon^{(\alpha\theta)/2}\big[g_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\big] : t \ge 0\big\}$ converges in probability toward 0, uniformly on compact sets.

Step 4) Our processes are valued in the Skorokhod space of càdlàg functions $\mathrm{D}([0,\infty))$ endowed with J_1 Skorokhod topology (see [46], §3.3). It is not difficult to see that a sequence which converges in probability toward 0, uniformly on compact sets, is also convergent in probability for J_1 metric, hence in distribution in J_1 topology. Recall that in the Skorokhod space, the summation is not a continuous map (see for instance [46], p. 84). In our case, the limits of the terms on the right hand side of equality (4.3.20) are, respectively 0 and a Brownian motion which have continuous paths. By using the joint convergence theorem (Theorem 11.4.5, p. 379 in [46]) and the continuous-mapping theorem (Theorem 3.4.3, p. 86 in [46]), we obtain the conclusion of Theorem 4.3.1. More precisely, the convergence in the theorem holds in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed by the uniform topology. Let us note that our situation is simpler than in [19] since the limit is a continuous paths process.

□

Proposition 4.3.5. *Assume that $\alpha \in (0,2)$ and $\beta + \frac{\alpha}{2} > 2$. The constant $\kappa_{\alpha,\beta}$ of the second part of Theorem 4.3.1, given in (4.3.21), satisfies*

$$\kappa_{\alpha,\beta} = -2\int_{\mathbb{R}} (x - \int x m_{\alpha,\beta}(\mathrm{d}x)) g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{4.3.22}$$

Proof. Since, by (4.3.19) and (4.3.21), $\kappa_{\alpha,\beta} = \frac{1}{t}\mathbb{E}\big[M_t^2\big]$, for all $t > 0$, by taking $t = \varepsilon^{\alpha\theta}$ and using (4.3.14), we get

$$\begin{aligned}
\kappa_{\alpha,\beta} &= \varepsilon^{-\alpha\theta}\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0) - \int_0^{\varepsilon^{\alpha\theta}} (V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathrm{d}s\Big)^2\Big] \\
&= \varepsilon^{-\alpha\theta}\Big\{\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\Big] + \mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} (V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathrm{d}s\Big)^2\Big] \\
&\quad - 2\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} (V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathrm{d}s\Big]\Big\}.
\end{aligned} \tag{4.3.23}$$

The first term on the right hand side of (4.3.23) can be written :

$$\begin{aligned}
\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\Big] &= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0) g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}})\Big] \\
&= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\mathbb{E}\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) \mid V_0\Big)\Big] \\
&= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\big(\mathcal{T}_{\varepsilon^{\alpha\theta}} g_{\alpha,\beta}\big)(V_0)\Big] \\
&= 2\int g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\Big(g_{\alpha,\beta}(V_0) + \int_0^{\varepsilon^{\alpha\theta}} \mathcal{T}_s(\mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_0)\mathrm{d}s\Big)\Big] \\
&= -2\mathbb{E}\Big[g_{\alpha,\beta}(V_0)\int_0^{\varepsilon^{\alpha\theta}} \mathcal{T}_s(\mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_0)\mathrm{d}s\Big] \\
&= -2\int g_{\alpha,\beta}(x)\, m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}} \mathcal{T}_s(\mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x))(x)\mathrm{d}s \\
&= -2\,\varepsilon^{\alpha\theta}\int (x - \int x m_{\alpha,\beta}(\mathrm{d}x)) g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) \\
&\quad - 2\int g_{\alpha,\beta}(x)\, m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}} \big((\mathcal{T}_s\mathrm{id}) - \mathrm{id}\big)(x)\mathrm{d}s.
\end{aligned}$$

By using the Hölder inequality, we prove that,

$$\mathbb{E}\left[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)^2\right] \sim -2\,\varepsilon^{\alpha\theta}\int (x - \int y m_{\alpha,\beta}(\mathrm{d}y)) g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x), \quad \text{as } \varepsilon \to 0. \tag{4.3.24}$$

We need to distinguish two cases following the position of β with respect to 2. Indeed, if $2 - \frac{\alpha}{2} < \beta < 2$,

$$g_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0} \|(\mathcal{T}_s\mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{3-\beta}(m_{\alpha,\beta})} = 0.$$

If $\beta \geq 2$,

$$\mathscr{g}_{\alpha,\beta} \in \mathrm{L}^{\frac{1}{\gamma}}(\mathscr{m}_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0} \|(\mathcal{T}_s \mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{1/(1-\gamma)}(\mathscr{m}_{\alpha,\beta})} = 0.$$

By using (4.3.17) and Fubini's theorem, the second term on the right hand side of (4.3.23) can be written

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} (V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathrm{d}s\Big)^2\Big] \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big((V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_u - \int x m_{\alpha,\beta}(\mathrm{d}x))\Big)\mathrm{d}u \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big((V_{s-u} - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_0 - \int x m_{\alpha,\beta}(\mathrm{d}x))\Big)\mathrm{d}u \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s \int_0^s \mathbb{E}\Big((V_0 - \int x m_{\alpha,\beta}(\mathrm{d}x))\, \mathcal{T}_{s-u}(\mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_0)\Big)\mathrm{d}u \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\, \mathbb{E}\Big((V_0 - \int x m_{\alpha,\beta}(\mathrm{d}x)) \int_u^{\varepsilon^{\alpha\theta}} \mathcal{T}_{s-u}(\mathrm{id} - \int x m_{\alpha,\beta}(\mathrm{d}x))(V_0)\,\mathrm{d}s\Big) \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\, \mathbb{E}\Big[(V_0 - \int x m_{\alpha,\beta}(\mathrm{d}x))\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}\big)(V_0) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)\Big] \\
&\quad = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u \int (x - \int y m_{\alpha,\beta}(\mathrm{d}y))\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}\big) - \mathscr{g}_{\alpha,\beta}\Big)(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x).
\end{aligned}$$

Once again by the Hölder inequality, we prove that

$$\mathbb{E}\left[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big)^2\right] = o(\varepsilon^{\alpha\theta}), \;\; \text{as } \varepsilon \to 0. \tag{4.3.25}$$

Indeed, if $2 - \frac{\alpha}{2} < \beta < 2$ then $\mathrm{id} \in \mathrm{L}^{3-\beta}(\mathscr{m}_{\alpha,\beta})$, and we note that

$$\lim_{\varepsilon\to 0} \sup_{0\leq u\leq \varepsilon^{\alpha\theta}} \|(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}) - \mathscr{g}_{\alpha,\beta}\|_{\mathrm{L}^{3-\beta/2-\beta}(\mathscr{m}_{\alpha,\beta})} = 0.$$

Similarly, if $\beta \geq 2$ then $\mathrm{id} \in \mathrm{L}^{1/(1-\gamma)}(\mathscr{m}_{\alpha,\beta})$, and we see that

$$\lim_{\varepsilon\to 0} \sup_{0\leq u\leq \varepsilon^{\alpha\theta}} \|(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}) - \mathscr{g}_{\alpha,\beta}\|_{\mathrm{L}^{\frac{1}{\gamma}}(\mathscr{m}_{\alpha,\beta})} = 0.$$

Finally, the third term in (4.3.23) is analysed by using the Cauchy-Schwarz inequality and the behaviour of the other terms. We get that

$$-2\,\mathbb{E}\Big[\Big(\mathscr{g}_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} (V_s - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathrm{d}s\Big] = o(\varepsilon^{\alpha\theta}), \;\; \text{as } \varepsilon \to 0. \tag{4.3.26}$$

Putting together (4.3.23)-(4.3.25), we obtain that

$$\kappa_{\alpha,\beta} = -2\int (x - \int x m_{\alpha,\beta}(\mathrm{d}x))\mathscr{g}_{\alpha,\beta}(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) + o(1), \;\; \text{as } \varepsilon \to 0.$$

and the result is proved. □

Chapter 5

Some extensions

5.1 Asymptotic stability of SDEs driven by the sum of the Brownian motion and a pure jump α-stable Lévy process

In this section, we want to study the asymptotic behaviour of the solution of a similar equation as in Chapter 4, starting from 0, but driven by a small noise which is the sum between the Brownian motion and a pure jump α-stable Lévy process. Consider

$$\begin{cases} v_t^1 = -\int_0^t \mathrm{sgn}(\mathrm{v_s^1})|\mathrm{v_s^1}|^\beta \mathrm{ds} + \varepsilon \mathrm{b_t} + \varepsilon \ell_\mathrm{t}, \quad \mathrm{v_0^1} = 0 \\ x_t^1 = \int_0^t v_s^1 \mathrm{d}s, \end{cases}$$

where ℓ is a α-stable Lévy process with $\alpha \in (0,2)$ (not totally asymmetric). We will compare this solution with the solution of the equation driven by a noise which is a pure jump α-stable Lévy process. We will show that the Brownian motion part has no contribution to the asymptotic behaviour. Introduce

$$\begin{cases} v_t^2 = -\int_0^t \mathrm{sgn}(\mathrm{v_s^2})|\mathrm{v_s^2}|^\beta \mathrm{ds} + \varepsilon \ell_\mathrm{t}, \quad \mathrm{v_0^2} = 0 \\ x_t^2 = \int_0^t v_s^2 \mathrm{d}s, \end{cases}$$

and perform the time change

$$V_t^1 = v_{\varepsilon^{-\alpha} t}^1 \quad \text{and} \quad \mathrm{X_t^1} = \mathrm{x}_{\varepsilon^{-\alpha}\mathrm{t}}^1$$

$$V_t^2 = v_{\varepsilon^{-\alpha} t}^2 \quad \text{and} \quad \mathrm{X_t^2} = \mathrm{x}_{\varepsilon^{-\alpha}\mathrm{t}}^2.$$

We obtain

$$\begin{cases} V_t^1 = -\varepsilon^{-\alpha} \int_0^t \mathrm{sgn}(\mathrm{V_s^1})|\mathrm{V_s^1}|^\beta \mathrm{ds} + \varepsilon^{1-\frac{\alpha}{2}} \mathrm{B_t} + \mathrm{L_t} \\ X_t^1 = \varepsilon^{-\alpha} \int_0^t V_s^1 \mathrm{d}s \end{cases}$$

$$\begin{cases} V_t^2 = -\varepsilon^{-\alpha} \int_0^t \mathrm{sgn}(\mathrm{V_s^2})|\mathrm{V_s^2}|^\beta \mathrm{ds} + \mathrm{L_t} \\ X_t^2 = \varepsilon^{-\alpha} \int_0^t V_s^2 \mathrm{d}s \end{cases}$$

with

$$B_t = \varepsilon^{\frac{\alpha}{2}} b_{\varepsilon^{-\alpha} t} \quad \text{and} \quad \mathrm{L_t} = \varepsilon \ell_{\varepsilon^{-\alpha}\mathrm{t}}.$$

Theorem 5.1.1. *Assume that $\beta + \frac{\alpha}{2} > 2$, and recall that $\theta := \frac{\alpha}{\alpha+\beta-1} \in (0,1)$. Then there exists a positive constant $\kappa_{\alpha,\beta}$ such that the process*

$$\left\{\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\Big(x^1_{\varepsilon^{-\alpha}t} - \varepsilon^{\theta-\alpha}t\int x m_{\alpha,\beta}(\mathrm{d}x)\Big) : t \geq 0\right\} \tag{5.1.1}$$

converges in distribution toward a Brownian motion with diffusion coefficient $\kappa_{\alpha,\beta}$, when $\varepsilon \to 0$. Here $m_{\alpha,\beta}$ is the invariant measure of the rescaled process of V^2 defined in the preceding Chapters 3 and 4.

Proof. The result has been already obtained for X^2 in Chapter 4, we only need to prove that $\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}\Big(X^1_t - X^2_t\Big)$ converges UCP toward 0 to conclude. The process $V^1 - V^2$ has continuous paths and verifies

$$\begin{aligned}|V^1_t - V^2_t|^2 = 2\varepsilon^{-\alpha}&\int_0^t (V^1_s - V^2_s)(-\mathrm{sgn}(\mathrm{V}^1_\mathrm{s})|\mathrm{V}^1_\mathrm{s}|^\beta + \mathrm{sgn}(\mathrm{V}^2_\mathrm{s})|\mathrm{V}^2_\mathrm{s}|^\beta)\mathrm{ds}\\ &\qquad + 2\varepsilon^{1-\frac{\alpha}{2}}\int_0^t (V^1_s - V^2_s)\mathrm{d}B_s + \varepsilon^{2-\alpha}t\\ &\qquad\qquad \leq 2\varepsilon^{1-\frac{\alpha}{2}}\int_0^t (V^1_s - V^2_s)\mathrm{d}B_s + \varepsilon^{2-\alpha}t,\end{aligned}$$

since $(V^1_s - V^2_s)(-\mathrm{sgn}(\mathrm{V}^1_\mathrm{s})|\mathrm{V}^1_\mathrm{s}|^\beta + \mathrm{sgn}(\mathrm{V}^2_\mathrm{s})|\mathrm{V}^2_\mathrm{s}|^\beta) \leq 0$. Using Gronwall's lemma, BDG's inequality and the fact that $a^2 \leq a^4 + 1$ for all $a \in \mathbb{R}$, we get

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|V^1_t - V^2_t|^4\Big) = \mathbb{E}\Big[\Big(\sup_{0\leq t\leq T}|V^1_t - V^2_t|^2\Big)^2\Big] \leq (2\varepsilon^{4-4\alpha}T^2 + 8c_2\varepsilon^{2-\alpha}T)\exp(8c_2\varepsilon^{2-\alpha}T).$$

We deduce that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|X^1_t - X^2_t|^4\Big) \leq \frac{T^4}{\varepsilon^{4\alpha}}\mathbb{E}\Big(\sup_{0\leq t\leq T}|V^1_t - V^2_t|^4\Big) \leq (2\varepsilon^{4-8\alpha}T^6 + 8c_2\varepsilon^{2-5\alpha}T^5)\exp(8c_2\varepsilon^{2-\alpha}T),$$

and the result follows. □

5.2 Some introduction to the dimension 2

In this section, we present a simple generalization of the equations studied in Chapters 3 and 4 in dimension 2. Again, we study the asymptotic behaviour of the solutions. We consider only the Brownian case:

$$\begin{cases} v^{\varepsilon,1}_t = \varepsilon b^1_t - \int_0^t \mathrm{sgn}(v^{\varepsilon,1}_s)|v^{\varepsilon,1}_s|^\beta \mathrm{d}s, & v^{\varepsilon,1}_0 = 0\\ v^{\varepsilon,2}_t = \varepsilon b^2_t - \int_0^t \mathrm{sgn}(v^{\varepsilon,2}_s)|v^{\varepsilon,2}_s|^\beta \mathrm{d}s, & v^{\varepsilon,2}_0 = 0\\ x^{\varepsilon,1}_t = \int_0^t v^{\varepsilon,1}_s \mathrm{d}s, & \\ x^{\varepsilon,2}_t = \int_0^t v^{\varepsilon,2}_s \mathrm{d}s, & \end{cases}$$

where $b_t := (b^1_t, b^2_t)$ is a 2-dimensional centered Brownian motion with covariance matrix $t\begin{pmatrix} 1 & \rho\\ \rho & 1\end{pmatrix}$, $\rho \in [-1,1]$ and $\beta \geq 1$. By using the self-similarity, we can see that the process $\{\mathcal{B}_t^\varepsilon := \varepsilon b_{\varepsilon^{-2}t} : t \geq 0\}$ is also a Brownian motion with the same distribution as b. Let us denote, for $t \geq 0$,

$$\mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad \mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon$$

satisfying, respectively,

$$\begin{cases} \mathcal{V}_t^{\varepsilon,1} = \mathcal{B}_t^{\varepsilon,1} - \frac{1}{\varepsilon^2}\int_0^t \mathrm{sgn}(\mathcal{V}_s^{\varepsilon,1})|\mathcal{V}_s^{\varepsilon,1}|^\beta \mathrm{d}s \\ \mathcal{V}_t^{\varepsilon,2} = \mathcal{B}_t^{\varepsilon,2} - \frac{1}{\varepsilon^2}\int_0^t \mathrm{sgn}(\mathcal{V}_s^{\varepsilon,2})|\mathcal{V}_s^{\varepsilon,2}|^\beta \mathrm{d}s \\ \mathcal{X}_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t \mathcal{V}_s^\varepsilon \mathrm{d}s. \end{cases}$$

To simplify the notations, we set

$$\theta := \frac{2}{\beta+1} > 0,$$

and we introduce

$$B_t^\varepsilon := \frac{\mathcal{B}_{t\,\varepsilon^{2\theta}}^\varepsilon}{\varepsilon^\theta} = \frac{b_{t\,\varepsilon^{(1-\beta)\theta}}}{\varepsilon^{(\beta-1)\theta/2}}, \quad \text{and} \quad V_t^\varepsilon := \frac{\mathcal{V}_{t\,\varepsilon^{2\theta}}^\varepsilon}{\varepsilon^\theta}.$$

Recall that

$$\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{2(2-\beta)}{(\beta+1)}} \int_0^{t\varepsilon^{-4/(\beta+1)}} V_s^\varepsilon \mathrm{d}s$$

and

$$\begin{cases} V_t^{\varepsilon,1} = B_t^{\varepsilon,1} - \int_0^t \mathrm{sgn}(V_s^{\varepsilon,1})|V_s^{\varepsilon,1}|^\beta \mathrm{d}s \\ V_t^{\varepsilon,2} = B_t^{\varepsilon,2} - \int_0^t \mathrm{sgn}(V_s^{\varepsilon,2})|V_s^{\varepsilon,2}|^\beta \mathrm{d}s. \end{cases}$$

The distribution of B^ε does not depend on ε so, to simplify the notation, we will suppress the index ε, as well as for V^ε.

All the results in one dimensional case stay true for V^1 and V^2 by considering each process separately. Recall that we set

$$\mathcal{g}_\beta(x) := \int_0^x \left(\int_y^{+\infty} -2z\mathrm{e}^{\mathcal{c}_\beta(z)}\mathrm{d}z\right)\mathrm{e}^{-\mathcal{c}_\beta(y)}\mathrm{d}y,\ x \in \mathbb{R},$$

with $\mathcal{c}_\beta(x) := -\frac{2}{\beta+1}|x|^{\beta+1}$. The infinitesimal generator of V is defined by,

$$\begin{aligned} \mathcal{A}_\beta f(x,y) = &-\mathrm{sgn}(x)|x|^\beta \frac{\partial f}{\partial x}(x,y) - \mathrm{sgn}(y)|y|^\beta \frac{\partial f}{\partial y}(x,y) \\ &+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x,y) + 2\rho\frac{\partial^2 f}{\partial x \partial y}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right], \quad \forall \beta \geq 1,\ \forall f \in \mathcal{C}^2(\mathbb{R}^2). \end{aligned}$$

Introduce, for all $(x,y) \in \mathbb{R}^2$, $h(x,y) = x^4 + y^4$. Since $\beta \geq 1$, there exist a constant $d > 0$ and a compact set C such that, for all $(x,y) \in \mathbb{R}^2$,

$$\mathcal{A}_\beta h(x,y) \leq -4h(x,y) + d\mathbb{1}_C.$$

By Theorem 6.1 in [31] p. 536, V is exponentially ergodic. There exists a unique invariant measure $m_\beta(\mathrm{d}x, \mathrm{d}y)$, with the marginal measure $m_\beta(\mathrm{d}x) := 2\mathrm{e}^{c_\beta(x)}\mathrm{d}x$ which we normalize by $\frac{1}{m_\beta(\mathbb{R})}$. Define

$$\kappa_{2,\beta} := \begin{pmatrix} \frac{1}{m_\beta(\mathbb{R})} \int_{\mathbb{R}} g'_\beta(x)^2 m_\beta(\mathrm{d}x) & \rho \int_{\mathbb{R}^2} g'_\beta(x) g'_\beta(y) m_\beta(\mathrm{d}x, \mathrm{d}y) \\ \rho \int_{\mathbb{R}^2} g'_\beta(x) g'_\beta(y) m_\beta(\mathrm{d}x, \mathrm{d}y) & \frac{1}{m_\beta(\mathbb{R})} \int_{\mathbb{R}} g'_\beta(y)^2 m_\beta(\mathrm{d}y) \end{pmatrix}. \tag{5.2.2}$$

Thanks to the result in Chapter 3, we already know that $\int_{\mathbb{R}} g'_\beta(x)^2 m_\beta(\mathrm{d}x) < \infty$ and by Cauchy-Swartz's inequality, we have

$$\int_{\mathbb{R}^2} g'_\beta(x) g'_\beta(y) m_\beta(\mathrm{d}x, \mathrm{d}y) \leq \frac{1}{m_\beta} \int_{\mathbb{R}} g'_\beta(x)^2 m_\beta(\mathrm{d}x) < \infty,$$

so $\kappa_{2,\beta}$ is well defined.

Theorem 5.2.1. *Assume that $\alpha = 2$, $\beta > -1$ and recall that $\theta = \frac{2}{\beta+1}$. There exists a matrix $\kappa_{2,\beta}$ such that the process*

$$\{\varepsilon^{(\beta-1)\theta} x^\varepsilon_{\varepsilon^{-2}t} : t \geq 0\} = \{\varepsilon^{(\beta-1)\theta} X^\varepsilon_t : t \geq 0\} \tag{5.2.3}$$

converges in distribution, as $\varepsilon \to 0$, in the space of continuous functions $\mathrm{C}([0, \infty), \mathbb{R}^2)$ *endowed by the uniform topology, toward a 2 dimensional Brownian motion process with covariance $\kappa_{2,\beta}$.*

Proof. By applying Itô's formula, we can see that

$$\begin{cases} g_\beta(V^1_t) = \int_0^t g'_\beta(V^1_s)\mathrm{d}B^1_s + \int_0^t V^1_s \mathrm{d}s \\ g_\beta(V^2_t) = \int_0^t g'_\beta(V^2_s)\mathrm{d}B^2_s + \int_0^t V^2_s \mathrm{d}s, \end{cases}$$

and therefore

$$\varepsilon^{(\beta-1)\theta} X^\varepsilon_t = \begin{pmatrix} -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^1_s)\mathrm{d}B^1_s + \varepsilon^\theta g_\beta(V^1_{t\,\varepsilon^{-2\theta}}) \\ -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^2_s)\mathrm{d}B^2_s + \varepsilon^\theta g_\beta(V^2_{t\,\varepsilon^{-2\theta}}) \end{pmatrix}.$$

For all $t \geq 0$, the continuous local martingale

$$M^\varepsilon_t := \begin{pmatrix} -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^1_s)\mathrm{d}B^1_s \\ -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^2_s)\mathrm{d}B^2_s \end{pmatrix}$$

has the quadratic variation

$$\langle M^\varepsilon \rangle_t = \begin{pmatrix} \varepsilon^{2\theta} \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^1_s)^2 \mathrm{d}s & \rho\varepsilon^{2\theta} \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^1_s) g'_\beta(V^2_s)\mathrm{d}s \\ \rho\varepsilon^{2\theta} \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^1_s) g'_\beta(V^2_s)\mathrm{d}s & \varepsilon^{2\theta} \int_0^{t\,\varepsilon^{-2\theta}} g'_\beta(V^2_s)^2 \mathrm{d}s \end{pmatrix}.$$

Since V is exponentially ergodic, for all $t \geq 0$,

$$\lim_{\varepsilon \to 0} \langle M^\varepsilon \rangle_t = \kappa_{2,\beta} t \text{ a.s.}$$

Using Whitt's theorem and the continuous mapping theorem, we obtain the result.

□

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